Journal of the Serbian Society for Computational Mechanics / Vol. 19 / No. 1, 2025 / pp 389-404 / review paper
(10.24874/jsscm.2025.19.01.24)

SOLVING WAVE ACOUSTIC EQUATION USING FINITE ELEMENT METHOD IN PAK SOLVER

Milica Nikolić[1,2,*] [0000-0002-8537-7944], Velibor Isailović[3] [0000-0002-1417-9633], Nenad Filipović[3,4,5] [0000-0001-9964-5615]

[1] Faculty of Mechanical and Civil Engineering in Kraljevo, University of Kragujevac, Dositejeva 19, 36000 Kraljevo, Serbia
e-mail: nikolic.mc@mfkv.kg.ac.rs
[2] Institute for information technologies Kragujevac, University of Kragujevac, Jovana Cvijića bb, 34000 Kragujevac, Serbia
e-mail: obradovicm@kg.ac.rs
[3] Faculty of Engineering, University of Kragujevac, Sestre Janjić 6, 34000 Kragujevac, Serbia
e-mail: velibor@kg.ac.rs
[4] Bioengineering Research and Development Center, Prvoslava Sojanovića 6, 34000 Kragujevac, Serbia
e-mail: fica@kg.ac.rs
[3] Science Fund of the Republic of Serbia, 27. marta, 39/II, 11000 Belgrade, Serbia
e-mail: fica@kg.ac.rs
**corresponding author*

Abstract

Wave acoustic equation is implemented and solved using finite element method in PAK solver to simulate propagation of sound waves, that cause oscillations of ear's components, such as cochlea (part of inner ear). To simulate mechanical behaviour of cochlea, behaviour of fluid in chambers and behaviour of basilar membrane which separates them have been analysed as one coupled system, since pressure changes in fluid directly affect movement of membrane. Therefore, strong fluid-solid coupling has been applied in common nodes on common surfaces, in a way that change in fluid pressure, its gradient in direction normal to the contact plane is equalised to the acceleration of basilar membrane nodes in contact, also normal to the common contact plane. This manuscript provides an overview of derivation of acoustic wave equation and its implementation into in-house built PAK solver, together with steps performed in fluid-solid coupling in simulation of mechanical cochlea model, followed by analysis of obtained results.

Keywords: acoustic wave equation, finite element method, PAK solver, fluid-solid coupling, mechanical cochlea models

1. Introduction

PAK is an in-house built solver written in Fortran programming language, that uses finite element method (FEM) for numerical solving of different problems in engineering area. It has been developed at Faculty of Mechanical Engineering, University of Kragujevac (today's Faculty of Engineering, University of Kragujevac) by the group of professors and assistants from this institution 50 years ago. PAK solver is able to provide a solution for solid and fluid domains and to capture fluid-solid interaction. The PAK source code has been upgraded throughout the years to enable various applications in engineering. Some of the PAK applications are simulation of lung epithelial cells barrier formation (Nikolic 2022, Nikolic et al. 2020), sedimentation process (Nikolic et al. 2021), behaviour of otoconia in semicircular canals (Vulovic et al. 2019), monocyte behaviour inside bioreactor (Nikolic et al. 2019, Nikolic et al. 2018), 3D modelling of plaque progression (Saveljic et al. 2018, Filipovic et al. 2017, Filipovic et al. 2012), simulation of blood flow and ablation process (Obradovic et al. 2010), numerical simulation of human hearing system (Isailovic et al. 2018, Isailovic et al. 2015, Nikolic et al. 2015, Isailovic et al. 2014).

This paper presents part of the research that has been performed in computationally analysed human hearing system, as a participation role in European horizon project Semantic Infostructure interlinking an open-source Finite Element tool and libraries with a model repository for the multi-scale Modelling and 3d visualization of the inner-ear (SIFEM), where primary work was development of computational models of inner ear. Inner ear has two major components – cochlea and semicircular canals. Cochlea is the part of inner ear in charge of detecting sounds, while semicircular canals have the main role in maintaining balance.

Several cochlea models have been developed during the project. The models differ by complexity, scales, geometry, outputs (mechanical and electrical response) (Nikolic 2017, Isailovic et al. 2016). Electrical models include Organ of Corti, placed within the cochlea, in scala media chamber, where transmission of mechanical movement to electrical signal occurs. These models are more complex in geometry and require smaller scale. On the other side, mechanical models of cochlea can be presented in more simple way. Tested geometries include coiled (Isailovic et al. 2015) and uncoiled version of cochlea (Nikolic et al. 2014). Uncoiled cochlea models were developed in different shape – box model (rectangular), conic shapes with straight and narrowing sides.

The simplest cochlea model is uncoiled, in shape of box, containing two fluid chambers divided by one membrane. More details can be found in Section 3 of this paper. External sound travels in waves from outer to the middle and finally inner ear. Oscillation of eardrum at the end of outer ear cause oscillation of three small bones in contact, placed in middle ear, that transmit these oscillations further to cochlea through oval window. Sound travels through fluid chambers causing changes in pressure and oscillations of basilar membrane. To be able to capture such a behaviour, acoustic wave equation is used to mathematically describe travelling of sound wave through cochlea. The equation is derived to the form suitable for numerical analysis by FEM in PAK solver.

Section 1 of this paper gives a brief overview of in-house developed PAK solver and some applications, focusing on one example of PAK utilisation in simulation of human hearing system and derivation of acoustic wave equation to capture phenomena of sound propagation through the medium. Details on derivation of acoustic equation and implementation in PAK solver are provided in Section 2. Section 3 gives more information of cochlea model and utilisation of derived acoustic wave equation. In Section 4, we present harmonic analysis and results obtained in PAK solver. The paper ends with final concluding remarks in Section 5.

2. Fundamental relations

Wave equation describes wave propagation through certain medium. These waves can be of different origin – light, sound, electromagnetic, etc. If the wave travelling through the media is sound wave, then wave equation is called acoustic wave equation. Wave equation belongs to the group of partial differential equations of second order, more specifically hyperbolic type of equation, considering its discriminant is less than zero (Kevorkian 1999). In general form, wave equation can be represented by Eq. 1 (Elliot et al. 2013)

$$\partial^2 u / \partial t^2 = c^2 \nabla^2 u \tag{1}$$

where c stands for speed of wave propagation that is constant for a specific medium, u is a scalar of interest, while $\nabla^2 = \Delta$ represents Laplacian.

Speed of wave can be in some cases dependent on wave frequency and that phenomenon calls dispersion. For such cases, wave speed c from Eq. 1 should be replaced with phase speed υ, as defined in Eq. 2.

$$\upsilon = \omega / k \tag{2}$$

In Eq. 2 k represents wave number and ω is an angular speed ($\omega = 2\pi f$).

If wave speed depends on amplitude of wave, then wave equation becomes nonlinear (Eq. 3).

$$\partial^2 u / \partial t^2 = c(u)^2 \nabla^2 u \tag{3}$$

In propagation of sound wave, a scalar variable of interest is acoustic pressure (p) that changes in space (Cartesian coordinates x, y and z) and time (t), as it is formulated in Eq. 4.

$$\partial^2 p / \partial t^2 = c^2 \nabla^2 p = c^2 (\partial^2 p / \partial x^2 + \partial^2 p / \partial y^2 + \partial^2 p / \partial z^2) \tag{4}$$

Acoustic wave equation can be solved analytically and numerically. Mostly used numerical methods for solving acoustic wave equation are finite difference method (FDM) (Smith 1985) and FEM (Grossmann et al. 2007, Morton and Mayers 2005). This paper presents derivation of acoustic wave equation for numerical solving in in-house built PAK solver using FEM.

2.1 Solving acoustic wave equation using FEM and PAK solver

Derivation starts from Eq. 4, division with c^2, putting everything on the left-hand side of the equation and volume integration (Fletcher 1984). We are using interpolation functions, N_k that are functions of space and not functions of time ($N_k = f(x_i), N_k \neq f(t)$)

$$\int_V N_k (\partial^2 p / \partial x^2 + \partial^2 p / \partial y^2 + \partial^2 p / \partial z^2) dV - 1/c^2 \int_V N_k (\partial^2 p / \partial t^2) dV = 0 \tag{5}$$

x_i for $i = 1,2,3$ stands for Cartesian coordinates x, y and z, respectively.

$$\int_V (N_k (\partial / \partial x_i)(\partial p / \partial x_i) - 1/c^2 N_k (\partial / \partial t)(\partial p / \partial t)) dV = 0 \tag{6}$$

Acoustic pressure can be written as a set of products of pressure values in element nodes (p^J) and corresponding interpolation functions (N_J).

$$p = N_J p^J \tag{7}$$

The first integral from Eq. 6, originated from Laplacian formulation, can be expressed through first derivative of product between interpolation function and first partial derivative of pressure (Eq. 8).

$$\int_V N_k (\partial / \partial x_i)(\partial p / \partial x_i) dV = \int_V (\partial / \partial x_i)(N_k \partial p / \partial x_i) dV - \int_V (\partial p / \partial x_i)(\partial N_k / \partial x_i) dV \tag{8}$$

Gauss's theorem (Stolze 1978) can be applied to the first member on right-hand side of the Eq. 8, so that integration by volume can be substituted with surface integration.

$$\iiint_V (\nabla \cdot F) dV = \oiint_S (F \cdot n) dS \tag{9}$$

Gauss's theorem (divergence theory) equalises volume integral with surface integral over a closed contour – surface that border considered volume. In that way, flow of vector field through the surface equalises with behaviour of vector flow within that surface. Unit vectors are directed from the surface. By applying Gauss's theorem (Eq. 9) to the first member on the right-hand side of Eq. 8 we obtain Eq. 10

$$\int_V (\partial / \partial x_i)(N_k \partial p / \partial x_i) dV = \int_S N_k (\partial p / \partial x_i) n_i dS \tag{10}$$

Substituting Eq. 7 and Eq. 10 into Eq. 8 leads to formulation provided in Eq. 11 and further Eq. 12.

$$\int_V N_k (\partial / \partial x_i)(\partial p / \partial x_i) dV = \int_S N_k (\partial p / \partial x_i) n_i dS - \int_V (\partial (N_J p^J) / \partial x_i)(\partial N_k / \partial x_i) dV \tag{11}$$

$$\int_V N_k (\partial / \partial x_i)(\partial p / \partial x_i) dV = \int_S N_k q_{Si} dS - \int_V (\partial N_k / \partial x_i)(\partial N_J / \partial x_i) p^J dV, q_{Si} = (\partial p / \partial x_i) n_i \tag{12}$$

Eq. 13 and Eq. 14 represents just further derivation of Eq. 12 and some substitution in annotating the members

$$\int_V N_k (\partial / \partial x_i)(\partial p / \partial x_i) dV = \int_S N_k q_{Si} dS - \int_V N_{k,i} N_{J,i} p^J dV, N_{k,i} = \partial N_k / \partial x_i, N_{J,i} = \partial N_J / \partial x_i \tag{13}$$

$$\int_V N_k (\partial / \partial x_i)(\partial p / \partial x_i) dV = F_k^S - p^J \int_V N_{k,i} N_{J,i} dV, F_k^S = \int_S N_k q_{Si} dS \tag{14}$$

F_k^S from Eq. 14 stands for surface force.

Next, we change obtained right-hand side of Eq. 14 into Eq. 6 to formulate Eq. 15 and Eq. 16.

$$F_k^S - p^J \int_V N_{k,i} N_{J,i} dV - 1/c^2 \int_V N_k (\partial / \partial t)(\partial (N_J p^J) / \partial t) dV = 0 \tag{15}$$

$$p^J \int_V N_{k,i} N_{J,i} dV + 1/c^2 \int_V N_k (\partial / \partial t)(\partial (N_J p^J) / \partial t) dV = F_k^S \tag{16}$$

Interpolation functions are not time dependent so they will not have derivation by time in contrast to pressure that will change with time. Taking this statement into account Eq. 16 becomes Eq. 17 and further Eq. 18 and Eq. 19.

$$p^J \int_V N_{k,i} \cdot N_{J,i} dV + 1/c^2 \int_V N_k N_J \ddot{p}^J dV = F_k^S \tag{17}$$

$$p^J \int_V N_{k,i} \cdot N_{J,i} dV + 1/c^2 \ddot{p}^J \int_V N_k N_J dV = F_k^S \tag{18}$$

$$p^J K_{kJ} + 1/c^2 \ddot{p}^J M_{kJ} = F_k^S \tag{19}$$

M_{kJ} and K_{kJ} from Eq. 19 stand for mass and stiffness matrix, respectively (Fahy and Gardonio 2006).

$$K_{kJ} = \int_V N_{k,i} \cdot N_{J,i} dV \tag{20}$$

$$M_{kJ} = \int_V N_k N_J dV \tag{21}$$

Upon performed derivation, acoustic wave equation has the form as presented with Eq. 22.

$$1/c^2 M_{kJ} \ddot{p}^J + K_{kJ} p^J = F_k^S \tag{22}$$

If we use natural boundary condition that external surface force is equal to zero ($F_k^S = 0$), acoustic wave equation has the final form (Eq. 23) (Kojic et al. 1998).

$$1/c^2 M_{kJ} \ddot{p}^J + K_{kJ} p^J = 0 \tag{23}$$

Therefore, starting from Eq. 6 we transformed acoustic wave equation to the Eq. 23 that is suitable for applying finite element method.

In order to numerically solve Eq. 23 Newmark's method was applied (Kojic et al. 1998). We used approximation that second order derivative of pressure in arbitrary time moment (τ) can be determined upon pressure values in two neighbouring time steps.

$$\ddot{p}(\tau) = (1-\delta)\,{}^t\ddot{p} + \delta\,{}^{t+\Delta t}\ddot{p} \tag{24}$$

In Eq.24 ${}^t\ddot{p}$ and ${}^{t+\Delta t}\ddot{p}$ stand for pressure values in time steps t and $t+\Delta t$ accordingly, while δ is a correction constant (usually used value is 0.5). Integration of Eq. 24 is performed on time interval Δt.

$$\dot{p}\,\big|_t^{t+\Delta t} = [(1-\delta)\,{}^t\ddot{p} + \delta\,{}^{t+\Delta t}\ddot{p}] \cdot t\,\big|_t^{t+\Delta t} \tag{25}$$

$${}^{t+\Delta t}\dot{p} - {}^t\dot{p} = [(1-\delta)\,{}^t\ddot{p} + \delta\,{}^{t+\Delta t}\ddot{p}] \cdot \Delta t \tag{26}$$

$${}^{t+\Delta t}\dot{p} = {}^t\dot{p} + [(1-\delta)\,{}^t\ddot{p} + \delta\,{}^{t+\Delta t}\ddot{p}] \cdot \Delta t \tag{27}$$

Another integration over time interval Δt is required due to second derivative of pressure.

$$p\,\big|_t^{t+\Delta t} = {}^t\dot{p} \cdot \Delta t + 1/2[(1-\delta)\,{}^t\ddot{p} + \delta\,{}^{t+\Delta t}\ddot{p}] \cdot \Delta t^2 \tag{28}$$

$${}^{t+\Delta t}p = {}^t p + {}^t\dot{p} \cdot \Delta t + 1/2[(1-\delta)\,{}^t\ddot{p} + \delta\,{}^{t+\Delta t}\ddot{p}] \cdot \Delta t^2 \tag{29}$$

For better expression of Eq. 29 coefficient δ is replaced with coefficient $\alpha = \delta/2$.

$${}^{t+\Delta t}p = {}^t p + {}^t\dot{p} \cdot \Delta t + (1/2-\alpha)\,{}^t\ddot{p} \cdot \Delta t^2 + \alpha\,{}^{t+\Delta t}\ddot{p} \cdot \Delta t^2 \tag{30}$$

$$\alpha\, {}^{t+\Delta t}\ddot{p} \cdot \Delta t^2 = {}^{t+\Delta t}p - {}^{t}p - {}^{t}\dot{p} \cdot \Delta t - (1/2 - \alpha)\, {}^{t}\ddot{p} \cdot \Delta t^2 \tag{31}$$

$${}^{t+\Delta t}\ddot{p} = (1/\alpha\Delta t^2)\, {}^{t+\Delta t}p - (1/\alpha\Delta t^2)\, {}^{t}p - (1/\alpha\Delta t)\, {}^{t}\dot{p} - ((1-2\alpha)/2\alpha)\, {}^{t}\ddot{p} \tag{32}$$

Once values of pressure at the end of time step and corresponding derivative values are established these expressions are substituted in Eq. 23.

$$M\ddot{p}^J + Kp^J = 0, M = 1/c^2 M_{kJ}, K = K_{kJ} \tag{33}$$

$$M[(1/\alpha\Delta t^2)\, {}^{t+\Delta t}p - (1/\alpha\Delta t^2)\, {}^{t}p - (1/\alpha\Delta t)\, {}^{t}\dot{p} - ((1-2\alpha)/2\alpha)\, {}^{t}\ddot{p}] + K\, {}^{t+\Delta t}p = 0 \tag{34}$$

$$M(1/\alpha\Delta t^2)\, {}^{t+\Delta t}p - M(1/\alpha\Delta t^2)\, {}^{t}p - M(1/\alpha\Delta t)\, {}^{t}\dot{p} - M((1-2\alpha)/2\alpha)\, {}^{t}\ddot{p} + K\, {}^{t+\Delta t}p = 0 \tag{35}$$

$$[M(1/\alpha\Delta t^2) + K]\, {}^{t+\Delta t}p = M[(1/\alpha\Delta t^2)\, {}^{t}p + (1/\alpha\Delta t)\, {}^{t}\dot{p} + ((1-2\alpha)/2\alpha)\, {}^{t}\ddot{p}] \tag{36}$$

$$\hat{K}\, {}^{t+\Delta t}p = M(a_0\, {}^{t}p + a_1\, {}^{t}\dot{p} + a_2\, {}^{t}\ddot{p}], \hat{K} = M(1/\alpha\Delta t^2) + K \tag{37}$$

a_i coefficients and $\hat{K}$ from Eq. 37 stand for:

$$\begin{aligned} a_0 &= 1/\alpha\Delta t^2 \\ a_1 &= 1/\alpha\Delta t \\ a_2 &= (1-2\alpha)/2\alpha \\ \hat{K} &= M(1/\alpha\Delta t^2) + K = M \cdot a_0 + K \end{aligned} \tag{38}$$

This derivation ensures incremental scheme of acoustic wave equation. In general way, we can find solution of Eq. 39 by time steps.

$$\hat{K}\, {}^{t+\Delta t}p = {}^{t+\Delta t}\hat{F}, {}^{t+\Delta t}\hat{F} = {}^{t+\Delta t}F + M(a_0\, {}^{t}p + a_1\, {}^{t}\dot{p} + a_2\, {}^{t}\ddot{p}], \hat{K} = M(1/\alpha\Delta t^2) + K \tag{39}$$

If solution cannot be achieved in incremental steps, we need to ensure correction of solution by iterations, meaning that solution obtain in time step $t + \Delta t$ and i iteration depends on pressure value in the same time step and previous iteration ($i-1$) enlarged for increment in value due to i^{th} iteration (Eq. 40).

$${}^{t+\Delta t}p^i = {}^{t+\Delta t}p^{i-1} + \Delta p^i \tag{40}$$

To calculate solution in current iteration we have to use a solution from previous iteration.

$${}^{t+\Delta t}\hat{K}^{i-1}\, {}^{t+\Delta t}p^i = {}^{t+\Delta t}\hat{F}^{i-1} \tag{41}$$

$${}^{t+\Delta t}\hat{K}^{i-1}({}^{t+\Delta t}p^{i-1} + \Delta p^i) = {}^{t+\Delta t}\hat{F}^{i-1} \tag{42}$$

$${}^{t+\Delta t}\hat{K}^{i-1}\, {}^{t+\Delta t}p^{i-1} + {}^{t+\Delta t}\hat{K}^{i-1}\Delta p^i = {}^{t+\Delta t}\hat{F}^{i-1} \tag{43}$$

$${}^{t+\Delta t}\hat{K}^{i-1}\Delta p^i = {}^{t+\Delta t}\hat{F}^{i-1} - {}^{t+\Delta t}\hat{K}^{i-1}\, {}^{t+\Delta t}p^{i-1} \tag{44}$$

Solving of Eq. 44 repeats until defined number of iterations or convergence criteria has been reached. Solution of first iteration is the solution from previous time step.

3. Utilisation of derived acoustic wave equation

Acoustic wave equation was used to simulate propagation of sound wave through cochlea, which represents part of the inner ear. Sound travels through outer ear, reaching eardrum and transmitting oscillation from eardrum to the section of middle ear (three small bones – hammer, anvil and stirrup). Middle ear is in contact with inner ear through oval window of cochlea. This is the section of cochlea that receives oscillations (originating from sound wave) that will be transmitted further through cochlea. Coiled shaped cochlea consists of three fluid chambers that are separated by two membranes. Three chambers filled with fluid are scala vestibuli, scala media and scala tympani. Scala vestibuli and scala media are separated by Reissner's membrane, while scala media and scala tympani are separated by basilar membrane. In simulating mechanical behaviour od cochlea, we can assume that scala media and Reissner's membrane can be omitted and take all into account as scala vestibuli. In that way, we obtain two fluid chambers – scala vestibuli and scala tympani divided by basilar membrane.

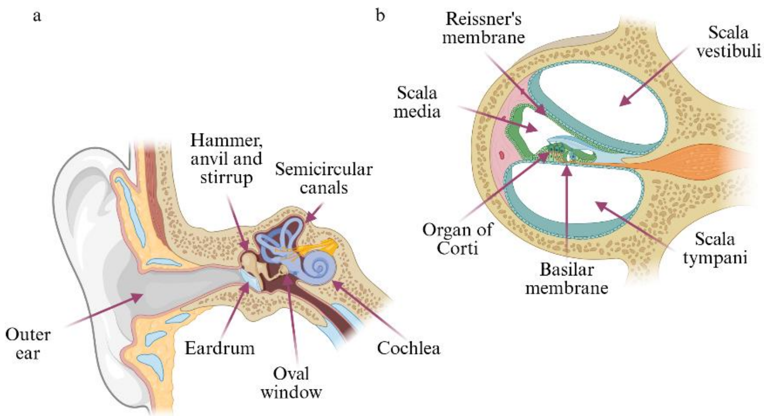

Fig. 1. Anatomy of ear. (a) Outer, middle and inner ear structures; (b) Cross-section of cochlea showing the scala vestibuli, scala media and scala tympani separated by two membranes - basilar and Reissner's membrane.

Sound wave travels through scala vestibuli, causing movement and oscillations of basilar membrane, influencing behaviour in scala tympani as well. Propagation of sound through fluid fulfilling scala vestibuli is represented by derived acoustic wave equation. Movement of basilar membrane, as a solid, is modelled with Newtonian dynamic equation (Kojic and Bathe 2005).

$$M_1\ddot{u} + B_1\dot{u} + K_1 u = F_1 \tag{45}$$

Metrices M_1, B_1 and K_1 stand for mass, damping and stiffness matrices respectively. $\ddot{u},\dot{u}$ and u are vectors of acceleration, velocity and displacement of nodes, while F_1 is vector of external force.

Numerical solution of Eq. 45 is performed with Newmark's method, like it was explained for acoustic wave equation (Kojic et al. 1998). Integration of Eq. 45 has been performed in N time steps. Upon derivation and forming of incremental-iterative scheme, we calculate displacement increment for i^{th} iteration and upon it, displacement at specific time step for i^{th} iteration.

$$ {}^{t+\Delta t}\hat{K}_1^{i-1}\Delta u^i = {}^{t+\Delta t}\hat{F}_1^{i-1} - {}^{t+\Delta t}\hat{K}_1^{i-1}\,{}^{t+\Delta t}u^{i-1} \quad (46) $$

To analyse together behaviour of sound propagation through fluid in cochlea chambers in line with caused oscillations of basilar membrane, two domains, fluid and solid, have to be coupled together. Domains can be coupled loosely or strongly (Filipovic et al. 2006). Loose coupling implies solving of domains alternately, meaning that solution from one domain at certain time step is used to correct geometrical parameters or boundary conditions of other domain before start solving that domain in that time step. Then solutions from other domain are returned to the first domain in the same manner before continuation of solving equations in next time step. Solutions obtain from loose coupled systems are usually less precise, but positive aspects of this approach are faster response and reduced computational resources. In the system is strongly coupled, both domains are solved simultaneously. To achieve simultaneous solving, coupling condition (coupling equation) has to be defined. Coupling equation is applied to the elements bordering two domains and this equation defines how domains affect each other.

In simulating mechanical behaviour of cochlea, we used strong coupling of domains by defining coupling equation to equalise solid and fluid forces in contact planes, orthogonal to the plane of contact and directed from common surface. Therefore, change of fluid pressure, its gradient in direction of normal to the common contact plane is equalised with acceleration of basilar membrane, also in direction normal to the common plane (Zienkiewicz 1983).

$$ n \cdot \nabla p = \rho n \cdot \ddot{u} \quad (47) $$

In Eq. 47 n stands for normal vector, ∇p is pressure gradient, ρ is density of basilar membrane. Coupling equation is implemented in PAK solver as boundary condition to form coupling matrices for fluid and solid.

Coupled system consisting of Eq. 33 for defining sound propagation through cochlea chambers and Eq. 45 for defining behaviour of basilar membrane. To make easier defining of coupling matrices, damping matrix from Eq. 45 is interpreted as complex component of stiffness matrix (Myklestadt 1952), so Eq. 45 can be rewritten in the form of Eq. 48 (with purpose to be similar in shape to acoustic wave equation).

$$ M_1\ddot{u} + K_1(1+i\eta)u = F_1 \quad (48) $$

Finally, coupled system for simulating mechanical behaviour of cochlea is presented with Eq. 49.

$$ \begin{bmatrix} M_1 & 0 \\ -\rho R & M \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} K_1(1+i\eta) & -S \\ 0 & K \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} F_1 \\ q \end{Bmatrix} \quad (49) $$

In Eq. 49 matrices R and S are coupling matrices formed upon coupling equation (Eq. 47) (Zienkiewicz 1983). There is a correlation between coupling matrices $R = S^T$, while q from acoustic wave equation stands for external excitation occurring at oval window, i.e. for nodes belonging to the oval window $q = 1$, otherwise it is equal to zero.

4. Harmonic analysis and results obtained in PAK solver

Coupled system defined with Eq. 49 can be analysed through harmonic analysis. Input in the system is harmonic function, in this case oscillations of nodes making the oval window. Output from a system has oscillatory character as well – wave propagation and oscillations of basilar membrane. In that way, we can assume that solutions of our coupled system have sinusoidal forms (Isailovic et al. 2014).

$$u = A_u \sin(\omega t + \alpha) \tag{50}$$

$$p = A_p \sin(\omega t + \alpha) \tag{51}$$

A_u and A_p are magnitudes of nodes displacement and pressure. In this analysis, we calculate first and second derivatives, as they are functioning in Eq. 49.

$$\dot{u} = A_u \omega \cos(\omega t + \alpha), \ddot{u} = -A_u \omega^2 \sin(\omega t + \alpha) \tag{52}$$

$$\dot{p} = A_p \omega \cos(\omega t + \alpha), \ddot{p} = -A_u \omega^2 \sin(\omega t + \alpha) \tag{53}$$

Next step is to substitute assumed solutions, first and second derivatives (Eq. 50 – 53) into Eq. 49. Additional assumption is that there is no external force acting on basilar membrane, so that F_1 is equal to zero.

$$\begin{bmatrix} K_1(1+i\eta) - \omega^2 M_1 & -S \\ -\rho R & K - \omega^2 M \end{bmatrix} \begin{Bmatrix} A_u \\ A_p \end{Bmatrix} = \begin{Bmatrix} 0 \\ q \end{Bmatrix} \tag{54}$$

Solving Eq. 54 comes down to solving characteristic modes of oscillation – eigen values and eigen vectors. For specified application and simulating mechanical behaviour of cochlea it is sufficient knowing modes of oscillations, rather than time and way of response.

Derived acoustic wave equation and dynamic equation of basilar membrane oscillations are implemented in in-house built PAK solver. Performed analysis and obtained results are presented below.

Mechanical cochlea model can be presented with two fluid chambers, scala vestibuli and scala tympani separated by basilar membrane, in coiled shape or unfold in rectangular or trapezoidal cross-section (box model). In box rectangular cochlea model due to symmetry only half of the geometry was created (Fig. 2) (Wang et al. 2012)

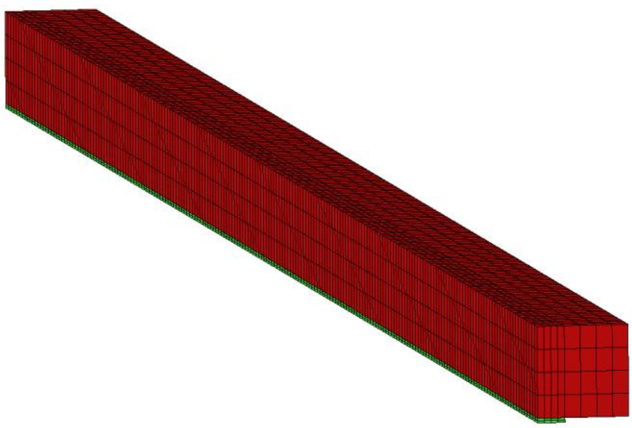

Fig. 2. Cochlea box model (uncoiled) – red section: scala vestibuli; green section: basilar membrane; applied geometrical symmetry to include behaviour of scala tympani below basilar membrane.

Young's modulus and damping coefficient change along the length of basilar membrane, while in other directions values are constant, meaning that basilar membrane can be considered as isotropic material (Ni 2012). When ear is stimulated by sound, basilar membrane oscillates. Each excitation frequency in the range human ear can detect (20Hz – 20 kHz) causes maximum oscillation and amplitude peak at exact position along 35 mm long basilar membrane (Greenwood 1961). Response of basilar membrane for external frequency of 1 kHz is presented in Fig. 3.

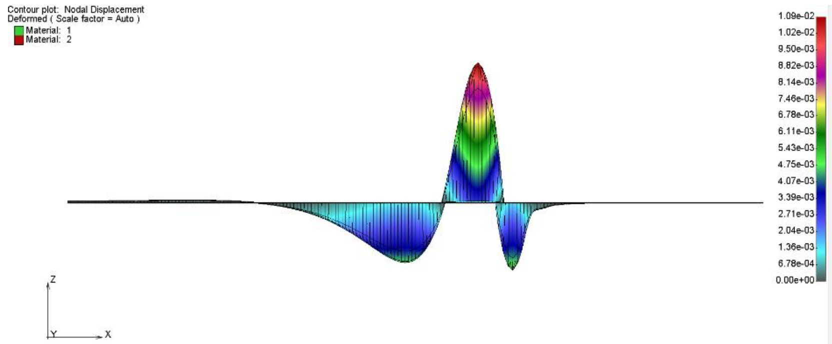

Fig. 3. Cochlea box model with longitudinal coupling – red section: scala vestibuli; green section: basilar membrane; applied geometrical symmetry.

Behaviour of cochlea can be presented through modal velocity – its magnitude and phase. Modal velocity has the form of harmonic function (e.g. sine, cosine functions) (Yoon et al. 2007). If we apply Euler's formula (Feynman et al. 1977), we can transform harmonic function into exponential function, containing magnitude and phase of modal velocity in expression (Fig. 4 and Fig. 5) (Nikolic 2017).

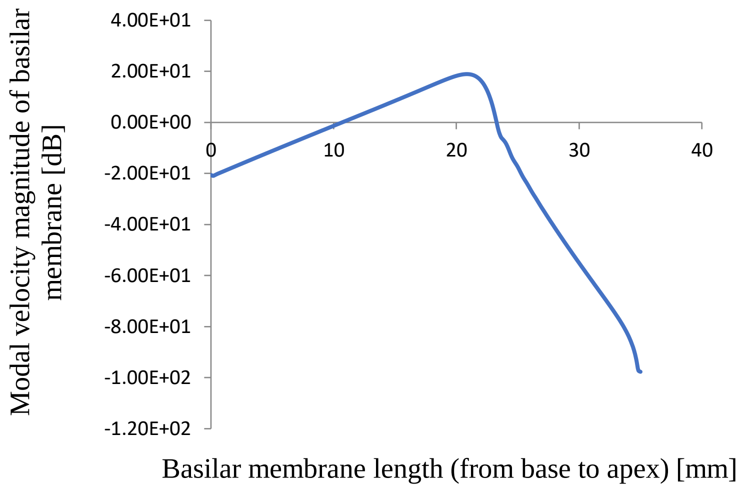

Fig. 4. Modal velocity magnitude of basilar membrane for excitation frequency of 1kHz in box uncoiled cochlea model with longitudinal coupling.

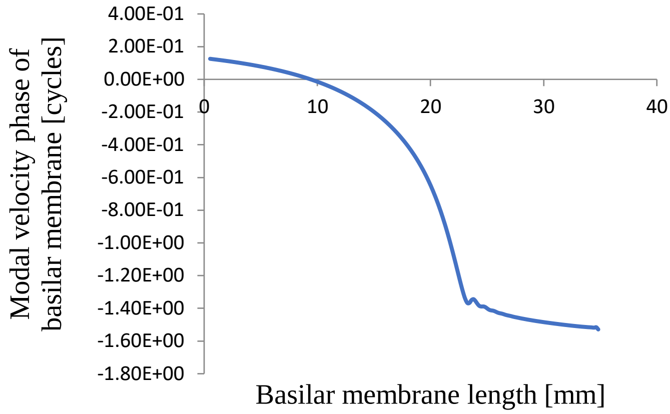

Fig. 5. Modal velocity phase of basilar membrane for excitation frequency of 1kHz in box uncoiled cochlea model with longitudinal coupling.

Derived mathematical models that was implemented in PAK solver had been used to analyse cochlea behaviour in conduction of sounds through air (air conduction, AC) and through bones (bone conduction, BC) (Isailovic et al. 2015b). Oscillations of basilar membrane when sound conducts through air (AC) and bones (BC) for excitation frequency of 1kHz and conic shape uncoiled cochlea model with narrowing sides are presented in Fig. 6.

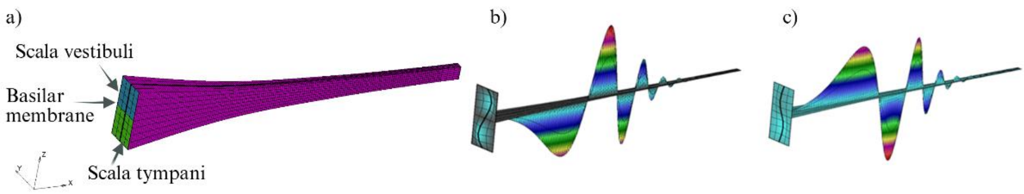

Fig. 6. (a) Geometry of narrowing cone-shaped uncoiled cochlea used for simulation of AC and BC; (b) Response of basilar membrane for AC and input frequency of 1 kHz; (c) Response of basilar membrane for BC and input frequency of 1 kHz.

In conducting sound through air, input to the model represents movement of stirrups in contact with scala vestibuli at oval window position, while in sound conduction through bones movement originates from surrounding bones in contact with cochlea, like temporal bone. Response of basilar membrane differs for AC and BC, but both models can precisely enough capture the position of peak corresponding to the input frequency.

We tested how much geometry of cochlea affects response from this mechanical model, and results showed small difference in obtained responses. Geometry has bigger influence in analysis of electrical components (Organ of Corti). Fig. 7 shows geometry of coiled cochlea and fluid pressure distribution inside of the fluid chambers for input frequency of 1 kHz.

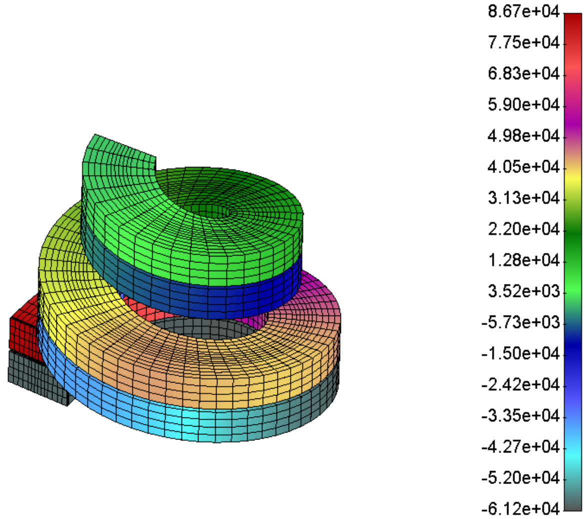

Fig. 7. Coiled cochlea model – obtained pressure distribution inside the fluid chambers for input frequency of 1 kHz.

Validation of obtained results from several mechanical cochlea models that have different geometries was performed by comparing obtained responses against Greenwood function (Greenwood 1961), that was created empirically (Fig. 8).

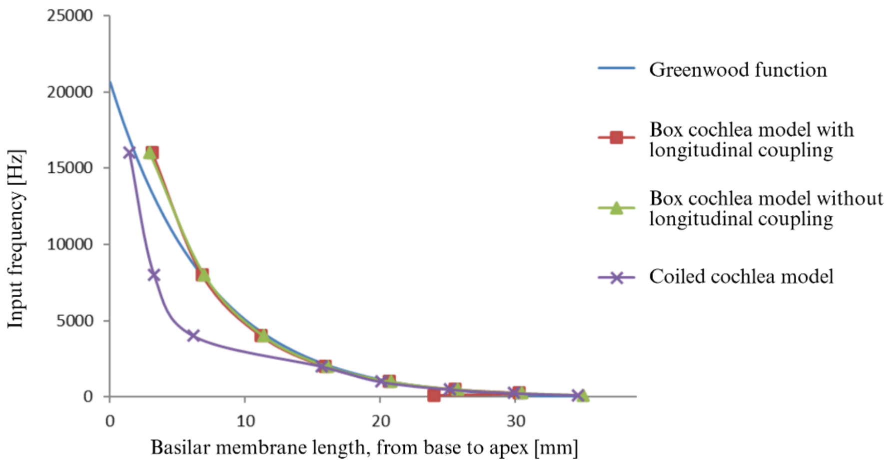

Fig. 8. Comparison of different cochlea models response for input frequency in range of human ear detecting sounds against empirically determined Greenwood function – relation of characteristic position at basilar membrane reaching maximum oscillation for corresponding input frequency.

Fig. 8 shows that all developed cochlea models for simulating mechanical behaviour follows appropriately Greenwood function. Box cochlea models with and without longitudinal coupling follow very well Greenwood function for input frequency up to 10 kHz. For higher values obtained results slightly diverge from empirically generated function. Coiled cochlea model shows good matching in obtained results for input frequency up to 2 kHz. Higher frequencies lead to bigger discrepancies but the trend is similar. In general, all mechanical cochlea models show good comparison with Greenwood function, with a slight variation in precision. Also, different tested geometries demonstrated that mechanical cochlea model response is not highly influenced by geometry, which is in accordance with literature (Manoussaki and Chadwick 2006). Derived numerical models and solutions obtained in PAK solver showed appropriate simulation of acoustic wave propagation applied for analysis of human hearing system and can be further used for additional analysis and more complex models.

4. Conclusions

This manuscript summarises efforts on deriving acoustic wave equation to the form suitable for numerical solving in in-house built PAK solver using FEM. Formed matrices and vectors are integrated into source code. The need for deriving and implementing acoustic wave equation in PAK solver arose from the need to recapitulate computationally behaviour of cochlea (part of inner ear). Further exploration on functioning of cochlea, including electrical and electro-mechanical models' development, was the subject of research question in European horizon project SIFEM, where authors of this manuscript participated.

Acknowledgements: The authors acknowledge support of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia, contract number [451-03-137/2025-03/200108 (Faculty of Mechanical and Civil Engineering in Kraljevo, University of Kragujevac)], contract number [451-03-136/2025-03/200378 (Institute for Information Technologies, University of Kragujevac)] and contract number [451-03-65/2024-03/200107 (Faculty of Engineering, University of Kragujevac)], and as well as FP7 ICT SIFEM project, number 600933. This research is in line with the strategic plan of the Republic of Serbia 2030, particularly in the section 2.2.2 Development of human resources, goal 3: Good health.

References

Elliot S J, Ni G, Mace B R and Lineton B (2013). A wave finite element analysis of the passive cochlea, The Journal of the Acoustical Society of America, Vol. 133, No. 3, pp. 1535-1545, ISSN 0001-4966

Fahy F and Gardonio P (2006). Sound and Structural Vibratio: Radiation, Transmission and Response, second edition, Elsevier Academic Press, ISBN: 9780123736338

Feynman R, Leighton R, Sands M (2006). The Feynman Lectures on Physics, Volume I, mainly mechanics, radiation, and heat, Addison-Wesley, ISBN 0-201-02010-6.

Filipovic N, Isailovic V, Milosevic Z, Nikolic D, Saveljic I, Radovic M, Nikolic M, Cirkovic-Andjelkovic B, Exarchos T, Fotiadis D, Pelosi G, Parodi O (2017). Computational modeling of plaque development in the coronary arteries, IFMBE Proceedings of the *International Conference on Medical and Biological Engineering, CMBEBIH 2017*, Sarajevo, Bosnia and Herzegovina, 16-18 March, 2017, Vol. 62, pp. 269-274, eISBN 978-981-10-4166-2, pISBN 978-981-10-4165-5, DOI:10.1007/978-981-10-4166-2_40

Filipovic N, Radovic M, Isailovic V, Milosevic Z, Nikolic D, Saveljic I, Milosevic M, Petrovic D, Obradovic M, Krsmanovic D, Themis E, Sakellarios A, Siogkas P, Marraccini P, Vozzi F, Meunier N, Teng Z, Fotiadis D, Parodi O, Kojic M (2012). Plaque formation and stent deployment with heating thermal effects in arteries, *Journal of the Serbian Society for*

Computational Mechanics, Vol. 6 No. 1, pp. 11-28, UDC: 004.921 ; 616.13-004.6-073, ISSN 1820-6530

Filipovic N, Mijailovic S, Tsuda A and Kojic M (2006). An Implicit Algorithm within the Arbitrary Lagrangian-Eulerian Formulation for Solving Incompressible Fluid Flow with Large Boundary Motions, Comp. Meth. Appl. Mech. Eng., Vol. 195, No. 44-47, pp. 6347-6361

Fletcher C A J (1984). Computational Galerkin Methods, Springer Series in Computational Physics, ISBN: 978-3-642-85951-9

Greenwood D D (1961). Critical Bandwidth and the Frequency Coordinates of the Basilar Membrane, The Journal of the Acoustical Society of America, Vol. 33, No. 10, pp. 1344–1356, ISSN 0001-4966

Grossmann C, Roos H, Stynes M (2007). Numerical Treatment of Partial Differential Equations, Springer Science & Business Media, ISBN 978-3-540-71584-9

Isailovic V, Nikolic M, Bibas T, Sakellarios A, Tachos N, Milosevic M, Filipovic N (2018). Numerical simulation of human hearing system, *2nd EAI International Conference on Future Access Enablers of Ubiquitous and Intelligent Infrastructures (FABULOUS 2016)*, Belgrade, Serbia, 24-25 October, 2016, Published in *EAI Endorsed Transactions on Pervasive Health and Technology*, Vol. 4, No. 13, e3, pp. 1-4, received on 06 June 2017, accepted on 14 November 2017, published on 28 February 2018, ISSN: 2411-7145, doi: 10.4108/eai.28-2-2018.154144

Isailovic V, Nikolic M, Nikolic D, Saveljic I, Filipovic N (2016). Using of Finite Element Method for Modeling of Mechanical Response of Cochlea and Organ of Corti, *ICIST 2016 6th International Conference on Information Society and Technology*, Kopaonik, Serbia, from 28 February to 2 March, 2016, Proceedings, Vol. 1, pp. 102-105, ISBN 978-86-85525-18-6

Isailovic V, Nikolic M, Milosevic Z, Saveljic I, Nikolic D, Radovic M, Filipovic N (2015). Finite Element Coiled Cochlea Model, *12th International Workshop on the Mechanics of Hearing*, Cape Sounio, Greece, 23-29 June, 2014. AIP Conference Proceedings 2015, Vol. 1703, No. 070015-1 – 070015-4, ISBN 978-0-7354-1350-4, ISSN 0094-243x, https://doi.org/10.1063/1.4939389

Isailovic V, Nikolic M, Milosevic Z, Saveljic I, Nikolic D, Radovic M, Filipovic N (2015b), Finite Element Model of Cochlea – Air Conduction and Bone Conduction, *ICIST 2015 5th International Conference on Information Society and Technology*, Kopaonik, Serbia, 8-11 March, 2015, Proceedings, Vol. 1, pp. 19-21, ISBN 978-86-85525-16-2

Isailovic V, Obradovic M, Nikolic D, Saveljic I, Filipovic N (2014). SIFEM project: Finite element modelling of the cochlea, *13th IEEE International Conference on Bioinformatics and Bioengineering (BIBE)*, Chania, Greece, 10-13 November, 2013, IEEE Xplore available online since 09.01.2014., pp. 1-4, eISBN 978-1-4799-3163-7, doi: 10.1109/BIBE.2013.6701611

Kevorkian J (1999). Partial Differential Equations: Analytical Solution Techniques, second edition, Texts in Applied Mathematics, Springer, ISBN-13: 978-0387986050

Kojic M and Bathe K J (2005). Inelastic Analysis of Solids and Structures, Springer, Berlin, Germany, ISBN 978-3-540-22793-9, 2005

Kojic M, Slavkovic R, Zivkovic M, Grujovic N (1998). Metod konacnih elemenata I – linearna analiza, Masinski fakultet u Kragujevcu, ISBN 86-80581-27-5

Manoussaki D and Chadwick R (2006). Effects of Geometry on Fluid Loading in a Coiled Cochlea, SIAM Journal on Applied Mathematics, Vol. 61, No. 2, pp. 369–386, ISSN (print): 0036-1399, ISSN (online): 1095-712X

Morton K W and Mayers D F (2005). Numerical Solution of Partial Differential Equations, second edition, Cambridge University Press, ISBN-13 978-0-521-60793-3

Myklestadt N O (1952). The concept of complex damping, Journal of Applied Mechanics, vol. 19, pp. 284-286

Ni G (2012). Fluid coupling and waves in the cochlea, PhD thesis, University of Southampton, Faculty of engineering and the environment, Institute of sound and vibration research, pp. 1-202

Nikolic M (2022). Chapter 7: Lab-on-a-chip for lung tissue from in silico perspective, in book Cardiovascular and Respiratory Bioengineering, Elsevier, pISBN: 978012823956807, eISBN: 9780128242292, Published: May 12, 2022, https://doi.org/10.1016/B978-0-12-823956-8.00007-9

Nikolic M, Sustersic T, Filipovic N (2021). Numerical Simulation of Sedimentation Process using Mason-Weaver Equation, 21st IEEE International Conference on BioInformatics and BioEngineering (BIBE), Kragujevac, Serbia (hybrid event), 25-27 October 2021, https://doi.ieeecomputersociety.org/10.1109/BIBE52308.2021.9635216

Nikolic M and Filipovic N (2020). Chapter 4 - Lung on a chip and epithelial lung cells modelling in book *Computational Modeling in Bioengineering and Bioinformatics*, pp. 105-135, ISBN 978-0-12-819583-3, https://doi.org/10.1016/B978-0-12-819583-3.00004-7

Nikolic M, Sustersic T, Muller C B, Zhang Y S, Vrana N E, Filipovic N (2019). Monocyte Behaviour under Perfusion Conditions for Development of Granuloma on-a-chip, *Tissue Engineering Therapies: From Concept to Clinical Translation & Commercialisation, TERMIS European Chapter Meeting 2019*, Rhodes, Greece, 27-31 May, 2019, Termis EU 2019 Conference Book, pp. 1356

Nikolic M, Sustersic T, Saveljic I, Vrana N E and Filipovic N (2018). Modelling of Monocytes Behaviour inside the Bioreactor, *Belgrade BioInformatics Conference, BelBi 2018*, Belgrade, Serbia, 18-22 June, 2018, Book of Abstracts, *Biologia Serbica*, Vol. 40, No. 1 (Special Edition), pp. 123, ISSN 2334-6590, UDK 57 (051)

Nikolic M (2017). Elektro-mehanicki model kohlee i analiza odziva modela (engl. Electro-mechanical cochlea model and model response analysis), doktorska disertacija, Fakultet inženjerskih nauka Univerziteta u Kragujevcu, datum odbrane: 19.09.2017., Kragujevac

Nikolic M, Teal P, Isailovic V, Filipovic N (2015). Finite Element Cochlea Box Model – Mechanical and Electrical Analysis of the Cochlea, *12th International Workshop on the Mechanics of Hearing*, Cape Sounio, Greece, 23-29 June, 2014. AIP Conference Proceedings 2015, Vol. 1703, No. 070012-1 – 070012-5, ISBN 978-0-7354-1350-4, ISSN 0094-243x, https://doi.org/10.1063/1.4939386

Obradović M, Avilla A, Thiagalingam A and Filipović N (2010). Finite element modeling of the endocardial radiofrequency ablation, *Journal of the Serbian Society for Computational Mechanics*, Vol. 4, No. 2, pp. 43-53, ISSN 1820-6530, UDC: 616.12-008.318-073:519.673

Saveljic I, Nikolic D, Milosevic Z, Isailovic V, Nikolic M, Parodi O and Filipovic N (2018). 3D Modeling of Plaque Progression in the Human Coronary Artery, *The 18th International Conference on Experimental Mechanics, ICEM 2018*, Brussels, Belgium, 1-5 July, 2018, Published by MDPI AG, Basel, Switzerland, Vol. 2, No 8, 388, pp.1-6, eISSN 2504-3900, doi:10.3390/ICEM18-05213

Smith G D (1985). Numerical Solution of Partial Differential Equations: Finite Difference Method, third edition, Oxford Applied Mathematics and Computing Science Series, ISBN 978-0198596509, 1985

Stolze C H (1978). A history of the divergence theorem, Historia Mathematica, Vol. 5, No. 4, pp. 437-442

Vulovic R, Nikolic M, Filipovic N (2019). Smart Platform for the Analysis of Cupula Deformation caused by Otoconia Presence within SCCs, *Computer Methods in Biomechanics and Biomedical Engineering*, Vol. 22, No. 2, pp. 130-138, ISSN 1476-8259, DOI 10.1080/10255842.2018.1539166

Wang X, Wang L, Zhou J and Hu Y (2012). Finite element modelling of human auditory periphery including a free-forward amplification of the cochlea”, Computer Methods in Biomechanics and Biomedical Engineering, Vol. 17, No. 10, pp. 1096-1107, doi 10.1080/10255842.2012.737458, 2012

Yoon Y, Puria S, Steele C R (2007). Frequency and Spatial Response of BasilarMembrane Vibration in a Three-Dimensional Gerbil Cochlear Model, Journal of Mechanics of Materials and Structures, Vol. 2, No. 8

Zienkiewicz O C (1983). The finite element method, third edition, Published by McGraw-Hill Book Co., New York, ISBN 10:0070840725 / ISBN 13:9780070840720