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INCREMENTAL LOAD STEP REGULATION FOR CONTACT MECHANICS USING THE PENALTY APPROACH

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Abstract

This paper presents a method for improving the numerical analysis of contact problems using a penalty formulation. Since contact phenomena are inherently nonlinear, particularly in the case of large deformations, an incremental solution strategy is required. Conventional Newton-Raphson algorithms often face convergence problems when many nodes make contact simultaneously. To overcome this limitation, we propose an automatic load step adjustment strategy based on a prediction-correction algorithm that limits the number of nodes that make contact per increment. The method is implemented in the finite element software PAK and tested on a reference pipe bending problem. The results show that the adaptive procedure reduces the number of increments required for convergence while maintaining the accuracy of the solution. This approach improves computational efficiency and robustness in finite element simulations of complex contact problems and provides a practical framework for addressing nonlinearities associated with friction and large deformations.

Keywords: Finite Element Method, Contact Algorithm, PAK (Program za Analizu Konstrukcija)

1. Introduction

The successful application of finite element solvers for contact problems requires significant expertise, as their general robustness and stability cannot be guaranteed. Contact analysis plays a key role in understanding the mechanical behavior at interfaces, thereby contributing to the improvement of structural reliability and safety (Grujovic N. (1996)). The work presented here introduces a framework for treating contact with friction by employing the penalty formulation (Vulovic S. (2008), Wriggers P. (2002)). This approach is advantageous because it is entirely geometry-based, eliminating the need to activate or deactivate additional degrees of freedom. Since frictional forces are irreversible, contact problems depend on the load history, which complicates the numerical treatment. To address this challenge, an automatic incrementation technique for applied loads has been incorporated into the computational algorithm. The increment size is determined by both the characteristics of the problem and the discretization of the contacting bodies.

Within this framework, a load scaling factor is computed at each potential contact node pair, and only the change corresponding to the smallest scaling factor is realized during an iteration. This ensures controlled progression of contact events. Although mathematical proof of uniqueness for general frictional contact cases is still lacking, the proposed models have been successfully implemented into the finite element software PAK (Kojic et al.). A numerical example illustrates the applicability and effectiveness of the developed algorithm in practical contact simulations.

2. Contact Kinematics

Contact may arise between a deformable body and a rigid obstacle, between two deformable bodies, or as self-contact. This study focuses on the case of two deformable bodies, $B^{(1)}$ and $B^{(2)}$, Fig. 1. Since the exact configuration of the interacting bodies is not known beforehand, contact inherently introduces nonlinearity, even when the bulk material follows linear elastic behavior (Vulovic S., 2008).

In standard contact mechanics notation, one surface is designated as the slave surface ($\Gamma_C^{(1)}$) and the other as the master surface ($\Gamma_C^{(2)}$). The non-penetration condition requires that no slave node may intrude into the master surface. The projection $\bar{\mathbf{x}}$ of a slave node $\mathbf{x}^k$ onto the current position of the master surface $\Gamma_C^{(2)}$, defined as:

$$\frac{\mathbf{x}^k - \bar{\mathbf{x}}(\bar{\xi}^1,\bar{\xi}^2)}{\left\|\mathbf{x}^k - \bar{\mathbf{x}}(\bar{\xi}^1,\bar{\xi}^2)\right\|}\cdot \bar{\mathbf{a}}_\alpha(\bar{\xi}^1,\bar{\xi}^2) = 0 \tag{1}$$

where $\alpha = 1,2$ and $\bar{\mathbf{a}}_\alpha(\bar{\xi}^1,\bar{\xi}^2)$ are the tangent covariant base vectors at the point $\bar{\mathbf{x}}$. These tangents are defined using the following relationships:

$$\bar{\mathbf{a}}_1 = \left.\frac{\partial \bar{\mathbf{x}}}{\partial \xi^1}\right|_{\xi^1=\bar{\xi}^1,\xi^2=\bar{\xi}^2}, \quad \bar{\mathbf{a}}_2 = \left.\frac{\partial \bar{\mathbf{x}}}{\partial \xi^2}\right|_{\xi^1=\bar{\xi}^1,\xi^2=\bar{\xi}^2}. \tag{2}$$

The relation (2) can be written as:

$$\bar{\mathbf{a}}_\alpha = \bar{\mathbf{x}}_{,\alpha}\left(\bar{\xi}^1,\bar{\xi}^2\right). \tag{3}$$

The definition of the projection point allows us to define the distance between any slave node and the master surface. The normal gap or the penetration g_N for slave, node *k* is defined as the distance between the current positions of this node to the master surface $\Gamma_C^{(2)}$:

$$g_N = (\mathbf{x}^k - \bar{\mathbf{x}})\cdot \bar{\mathbf{n}} \tag{4}$$

where $\bar{\mathbf{n}}$ refers to the normal to the master face $\Gamma_C^{(2)}$ at point $\bar{\mathbf{x}}$ (Fig. 1). Normal to be defined using tangent vectors at the point $\bar{\mathbf{x}}$ is:

$$\bar{\mathbf{n}} = \frac{\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2}{\left\|\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2\right\|}. \tag{5}$$

This gap (4) gives the non-penetration conditions as follows:

$$g_N = 0 \;\text{ perfect contact;} \qquad g_N > 0 \;\text{ no contact;} \qquad g_N < 0 \;\text{ penetration} \tag{6}$$

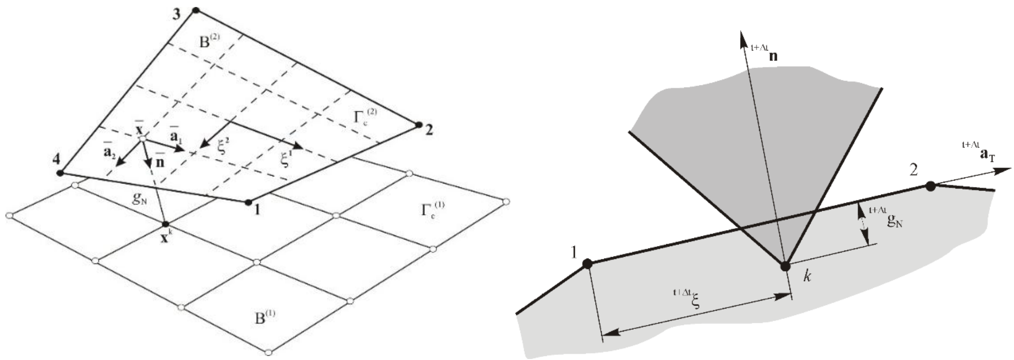

Fig. 1. Geometry of the 3D and 2D node-to-segment contact element

For frictionless contact, this non-penetration condition fully characterizes the interaction.

In the presence of friction, however, tangential relative displacements must also be considered. The sliding path of the slave node $\mathbf{x}^k$ across the master surface $\Gamma_C^{(2)}$ is expressed through the accumulated tangential relative displacement, in time interval from t_0 to t, as:

$$g_T = \int_{t_0}^{t} \|\dot{\mathbf{g}}_T\| dt = \int_{t_0}^{t} \left\| \dot{\bar{\xi}}^{\alpha} \bar{\mathbf{a}}_{\alpha} \right\| dt = \int_{t_0}^{t} \sqrt{\dot{\bar{\xi}}^{\alpha} \dot{\bar{\xi}}^{\beta} a_{\alpha\beta}}\, dt \tag{7}$$

In the geometrically linear case, the relative tangential velocity at the contact point is:

$$\dot{\mathbf{g}}_T = \dot{\bar{\xi}}^{\alpha} \bar{\mathbf{a}}_{\alpha} = \dot{g}_{T\alpha} \bar{\mathbf{a}}^{\alpha} \tag{8}$$

where is

$$\bar{a}_{\beta\alpha} \dot{\bar{\xi}}^{\beta} = \left[\dot{\mathbf{x}}^k - \dot{\bar{\mathbf{x}}} \right] \cdot \bar{\mathbf{a}}_{\alpha} = \dot{g}_{T\alpha} \tag{9}$$

and $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_{\alpha} \cdot \bar{\mathbf{a}}_{\beta}$ is the metric tensor in point $\bar{\mathbf{x}}$ of the master surface $\Gamma_C^{(2)}$.

2.1 Adjustment of the load step

When too many nodes establish contact within the same increment, numerical convergence becomes problematic. Ideally, only a limited number of nodes should enter contact at each load step. To achieve this, the proposed method adjusts the step size by introducing a prediction–correction algorithm.

The number of activated contact nodes N_c per step is defined by the user. The increase or the decrease of the load step is written as:

$$\Delta t_{cor} = \beta \Delta t \tag{10}$$

where β is a multiplicative factor which must be determined. A multiplicative correction factor is computed so that only the prescribed number of nodes is allowed to contact in the updated increment. With the goal to simplify and clearly describe the determination of β factor, we suppose that the reference face unchanged between times t and $t+\Delta t$ and that node remains in contact with the same face. The new load step is determined so that the considered node comes exactly into contact at time $t+\Delta t_{cor}$. So, ${}^{\Delta t_{cor}} g_N^k = 0$. For each node k, likely to come into contact, the factor β^k is defined by:

$$\beta^k = \frac{{}^t g_N^k}{{}^t g_N^k - {}^{t+\Delta t} g_N^k} \tag{11}$$

The scalar β^k is calculated at the first Newton iteration of the time $t+\Delta t$ for a given node likely to come into contact. By considering all the nodes, β can be calculated. Then, the load step is corrected by use of β factor. The calculation is then taken again with this new load step Δt_{cor} and so the configuration at the time $t+\Delta t_{cor}$ is evaluated. All the coeficients β^k are classified in an ascending order:

$$\beta^1 \le \beta^2 \le ... \le \beta^{N_c} \tag{12}$$

The aim of the proposed method is to limit the number of nodes coming into contact to N_c. So, the coefficient obtained by the relation (12) is:

$$\beta = \beta^{N_c} \tag{13}$$

3. Constitutive Equations for the Contact Interface

The contact stress vector at the interface is decomposed into normal and tangential components.

$$\bar{\mathbf{t}} = \bar{\mathbf{t}}_N + \bar{\mathbf{t}}_T = \bar{t}_N \bar{\mathbf{n}} + \bar{t}_{T\alpha} \bar{\mathbf{a}}^\alpha \tag{14}$$

where $\bar{\mathbf{a}}^\alpha$ is a contravariant base vector. The stress acts on both surfaces according to the action-reaction principle: $\bar{\mathbf{t}}(\bar{\xi}^1, \bar{\xi}^2) = -\mathbf{t}$ in the contact point $\bar{\mathbf{x}}$. The tangential stress $\bar{t}_{T\alpha}$ is zero in the case of frictionless contact. In the frictionless case, tangential stress vanishes, while in frictional contact both components must satisfy the Kuhn–Tucker conditions:

$$g_N \ge 0, \quad \bar{t}_N \le 0, \quad \bar{t}_N g_N = 0 \tag{15}$$

Within the penalty method, the normal stress is expressed as the product of the penalty parameter and the penetration value

$$t_N = \varepsilon_N g_N \tag{16}$$

In the tangential direction, two cases are distinguished:

- Stick condition, where no relative sliding occurs, and a linear penalty model describes the tangential stress

$$t_{T\alpha}^{stick} = \varepsilon_T g_{T\alpha} \tag{17}$$

- Slip condition, which arises once the tangential traction exceeds the frictional limit governed by Coulomb’s law.

$$t_{T\alpha}^{sl} = -\mu \left| \mathbf{t}_N \right| \frac{\dot{g}_{T\alpha}^{sl}}{\left\| \dot{\mathbf{g}}_T^{sl} \right\|} \tag{18}$$

To identify whether stick or slip occurs, an indicator function *f* is evaluated at each step, with respect the Coulomb's model for frictional interface law (Vulovic et al. 2007)

$$f = \left\| \mathbf{t}_T \right\| - \mu \left| t_N \right| \tag{19}$$

In equation (19) the first term is $\left\| \mathbf{t}_T \right\| = \sqrt{t_{T\alpha} \bar{a}^{\alpha\beta} t_{T\beta}}$.

A backward Euler integration scheme combined with a return-mapping procedure is applied to enforce the friction law. If a state of the stick is assumed, the trial values of the tangential contact pressure vector $t_{T\alpha}$, and the indicator function f at load step *n+1* can be expressed in terms of their values at load step *n* as follows:

$$t_{T\alpha\; n+1}^{trial} = t_{T\alpha\; n} + \varepsilon_T \Delta g_{T\alpha\; n+1} = t_{T\alpha\; n} + \varepsilon_T \bar{a}_{\alpha\beta} \Delta \xi_{n+1}^{\beta} \tag{20}$$

$$f_{Tn+1}^{trial} = \left\| \mathbf{t}_{Tn+1}^{trial} \right\| - \mu \left| t_{Nn+1} \right| \tag{21}$$

The return mapping is completed by:

$$t_{T\alpha\; n+1} = \begin{cases} t_{T\alpha\; n+1}^{trial} & \text{if } f \leq 0 \\ \mu \left| t_{Nn+1} \right| n_{T\alpha\; n+1}^{trial} & \text{if } f > 0 \end{cases} \tag{22}$$

with:

$$n_{T\alpha\; n+1}^{trial} = \frac{t_{T\alpha\; n+1}^{trial}}{\left\| \mathbf{t}_{Tn+1}^{trial} \right\|} \tag{23}$$

In physical terms, the penalty method can be interpreted as a system of elastic springs, which restore the bodies to the contact surface when overlap or sliding occurs.

3.1 Algorithm for frictional contact

For the solution a nonlinear equilibrium equation with inequality constraints (4) as a result of contact, we use a standard implicit method. In order to apply Newton's method for the solution system of the equilibrium equation, a linearization of the contact contributions is necessary. Linearization of the contact contributions yields tangent stiffness matrices for the normal, stick, and slip cases.

The algorithm of the automatic adjustment of load step for the frictional contact algorithm using the penalty method is shown in Table 1. At each iteration, penetration is evaluated, stick or slip conditions are determined through trial states, and the tangent matrices are updated accordingly. If excessive contact activations occur, the algorithm automatically reduces the step size and restarts the iteration.

LOOP over all load step
LOOP over all iterations
LOOP over all contact segment k
Determination of the penetration g_N^k
IF $i=1$ AND $I_{cor}=1$ THEN
Determination of β^k
(check for contact) IF $g_N \leq 0$ THEN
compute matrix $\mathbf{K}_N^e$ and ${}^{t+\Delta t}t_N$
(the first iteration) IF $i=1$ THEN
set all active nodes to state stick,
$\mathbf{t}_{Tn+1}$, compute matrix $\mathbf{K}_T^{stick}$
ELSE
Compute trial state: $t_{T\alpha\ n+1}^{trial}$ and f_{Tn+1}^{trial}
IF $f_{Tn+1}^{trial} \leq 0$ THEN
$t_{T\alpha\ n+1} = t_{T\alpha\ n+1}^{trial}$, compute matrix $\mathbf{K}_T^{stick}$
GO TO (a)
ELSE
$t_{T\alpha\ n+1} = \mu \left| t_{Nn+1} \right| n_{T\alpha\ n+1}^{trial}$, compute matrix $\mathbf{K}_T^{slip}$
ENDIF
ENDIF
ENDIF
IF ($i=1$ AND $I_{cor}=1$) GO TO (b)
ENDIF
(a) END LOOP
(b) END LOOP
IF $I_{cor}=0$ THEN
Correction of load step $\Delta t_{cor} = \beta \Delta t$
Update $I_{cor}=1$
Automatic restart with $\Delta t = \Delta t_{cor}$
ENDIF
END LOOP

Table 1. Algorithm of the automatic adjustment of load step for frictional contact algorithm using the penalty method

The tangent stiffness matrix for the normal contact is:

$$\mathbf{K}_N = \varepsilon_N \mathbf{N}\mathbf{N}^T \tag{24}$$

The symmetric tangent stiffness matrix for stick condition is:

$$\mathbf{K}_T^{stick} = \varepsilon_T \bar{a}_{\alpha\beta} \mathbf{D}^{\alpha} \mathbf{D}^{\beta T} \tag{25}$$

The tangent stiffness matrix for slip condition is:

$$\mathbf{K}_T^{slip} = \mu \varepsilon_N n_{T\alpha\ n+1}^{trial} \mathbf{D}^{\alpha} \mathbf{N}^T + \frac{\mu \varepsilon_N g_{Nn+1}}{\left\| \mathbf{t}_{Tn+1}^{trial} \right\|} \varepsilon_T \bar{a}_{\beta\gamma} \left[\delta_{\alpha}^{\beta} - n_{T\alpha\ n+1}^{trial} n_{T\ n+1}^{trial\ \beta} \right] \mathbf{D}^{\alpha} \mathbf{D}^{\gamma T} \tag{26}$$

where

$$\mathbf{N} = \begin{Bmatrix} \bar{\mathbf{n}} \\ -H_1\bar{\mathbf{n}} \\ -H_2\bar{\mathbf{n}} \\ -H_3\bar{\mathbf{n}} \\ -H_4\bar{\mathbf{n}} \end{Bmatrix} \quad \mathbf{D}^\alpha = \bar{a}^{\alpha\beta}\mathbf{T}_\beta \quad \mathbf{T}_\beta = \begin{Bmatrix} \bar{\mathbf{a}}_\beta \\ -H_1\bar{\mathbf{a}}_\beta \\ -H_2\bar{\mathbf{a}}_\beta \\ -H_3\bar{\mathbf{a}}_\beta \\ -H_4\bar{\mathbf{a}}_\beta \end{Bmatrix} \tag{26}$$

The linearization of $n_{T\alpha\ n+1}^{trial}$ gives (for details see Kojić & Bathe, 2005):

$$\Delta\left(n_{T\alpha\ n+1}^{trial}\right) = \Delta\left(\frac{t_{T\alpha\ n+1}^{trial}}{\left\|\mathbf{t}_{Tn+1}^{trial}\right\|}\right) = \frac{1}{\left\|\mathbf{t}_{Tn+1}^{trial}\right\|}\left[\delta_\alpha^\beta - n_{T\alpha\ n+1}^{trial} n_{T\ n+1}^{trial\ \beta}\right]\Delta t_{T\beta\ n+1}^{trial} \tag{27}$$

Notably, the stiffness matrix under slip conditions becomes non-symmetric due to the non-associative nature of Coulomb friction.

6. Example

A benchmark problem of tube bending is analyzed to demonstrate the performance of the method. A quarter of the tube is modeled using enhanced 4-node plane strain elements. The internal tube radius is $\mathrm{r} = 100\mathrm{mm}$ and thickness $\mathrm{t} = 20\mathrm{mm}$, elastic module $\mathrm{E} = 400\mathrm{MPa}$ and Poasson's ratio $\nu = 0.25$. The tube is modeled with 1313 elements – plain strain. Banding of the tube is conducted using a rigid plate, Fig. 2. Penalty parameter is $\varepsilon_{\mathrm{N}} = 1\times10^9$. The solution is obtained by 47 steps of displacement increments equal to 1.6 mm, Fig. 3.

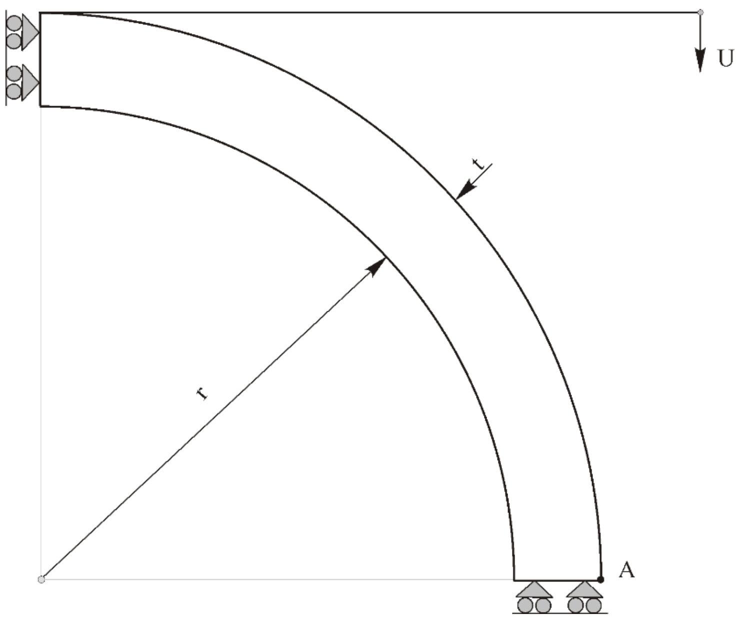

Fig. 2. Geometry of the model

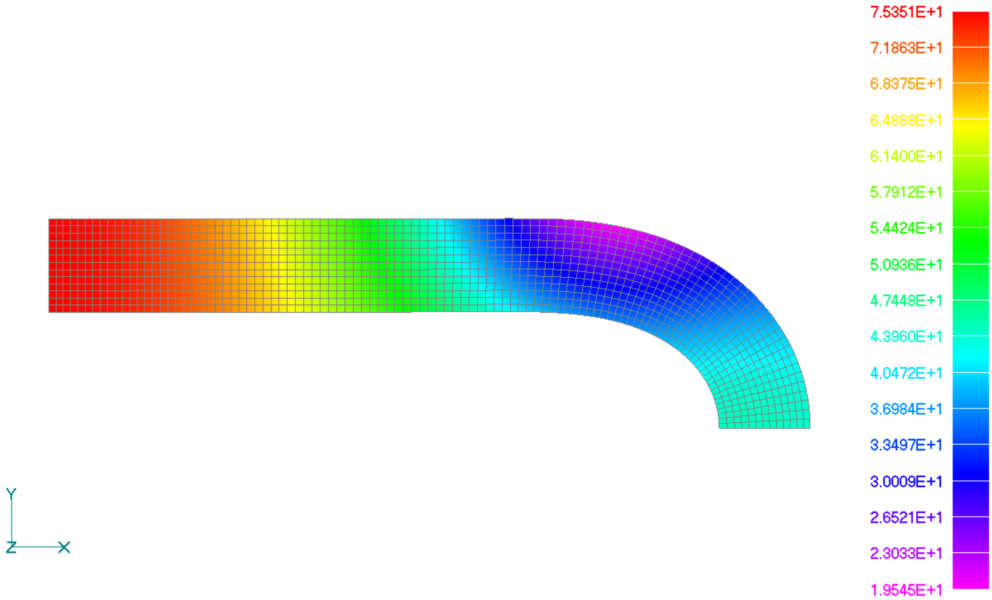

Fig. 3. Total displacement field, deformed configuration

The test is realized with automatic adjustment of the step in the case of contact. The data of this adjustment is $N_c = 2$. This example illustrates the utility of the method when the load step proposed by the user is too small. Without adjustment, 47 steps are necessary to obtain the imposed load. With the presented method, one can note a very strong increase in the load step, and therefore the number of steps necessary is reduced to 32. Fig. 4 illustrates this phenomenon and emphasizes a non-linear evolution of the load step.

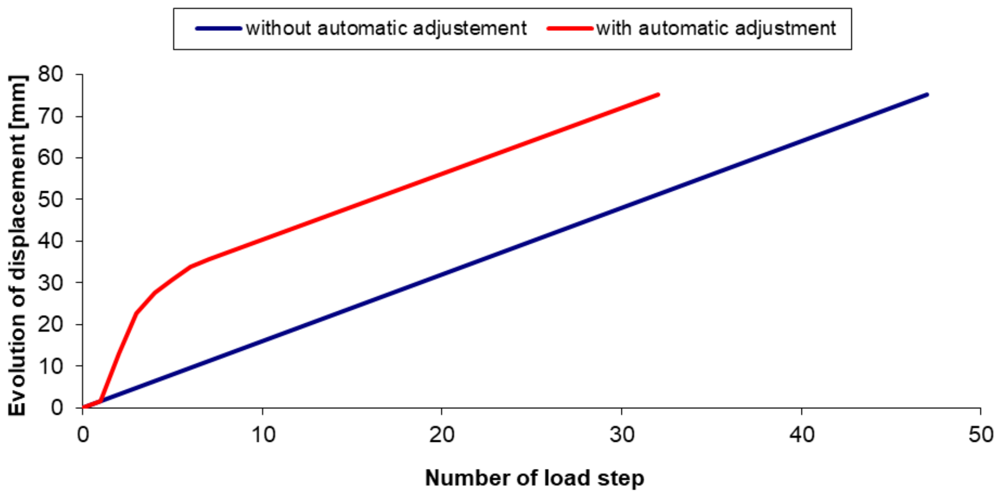

Fig. 4. Evolution of load step

7. Conclusions

This paper introduces a method for automatic adjustment of load increments in contact simulations based on the penalty formulation. By controlling the number of nodes that can change contact status in a single step, the approach improves convergence and efficiency of nonlinear finite element analyses. The numerical example demonstrates that the proposed algorithm can substantially reduce computational effort without compromising accuracy, making it a valuable tool for problems involving finite deformations and complex contact interactions.

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