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# IMPLICIT STRESS INTEGRATION AND GOVERNING PARAMETER METHOD (GPM) AS A GENERAL CONCEPT FOR INELASTIC MATERIAL MODELS IN THE CODE PAK

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## Abstract

In this paper, we give a short overview of the basic concept of stress integration for inelastic material models. This methodology was introduced by the author, initially for thermo-elastic-plastic and creep deformation of metals, termed as 'effective-stress-function' (ESF), and then generalized to the 'governing parameter method'. These formulations were further implemented in several geological material models. The author built this methodology into the finite element program ADINA, while they were introduced into the PAK program by other PAK authors, after publications in the relevant journals given here in the list of references. This methodology has also been extended to other material models cited here in the references, and served as the basic concept for these models. Only a few simple examples are borrowed from our previous publications since numerous applications of the PAK code over decades to real engineering problems are available in references, in various journals, reports, and books; some of these are given in this monograph.

**Keywords**: Materially nonlinear problems, implicit stress integration, governing parameter method, viscoplastic material model, PAK finite element program

## 1. Introduction

The main tasks in any nonlinear problem can be summarized as (Bathe, 1996; Kojic and Bathe, 2005): generality, accuracy, and efficiency. We will refer in the presentation of our methodology to the generality and efficiency, while regarding the accuracy, it can be stated, following findings in numerous references, that it was concluded that the implicit stress integration is favorable with respect to the explicit or mixed formulations. Namely, the use of the parameters and variables at the end of time (load) steps gives not only the most accurate results but also provides the best convergence rate. This approach is implemented in all material models built within our FE code PAK.

Our governing parameter method (GPM) assumes the condition that there is a parameter that governs the inelastic deformation within a time or load step during FE incremental-iterative computational procedure. This condition relies on the physical basis that there is positive power dissipation during inelastic material deformation; a detailed analysis of this is discussed in (Kojic and Bathe, 2005). The GPM represents a generalization of the 'Effective-Stress-Function' (ESF) introduced by the author in (Kojic and Bathe, 1987). In the case of the time-independent plasticity with a general form of the yield condition, or metal plasticity, viscoplasticity, and thermo-plasticity and creep, the governing parameter is the increment of the modulus of the increment of plastic strain or the increment of the effective plastic strain. On the other hand, the governing parameter for geological materials, such as cap models or Cam-clay model, or generalized geological model, stress integration is achieved according to the GPM. Details are given in (Kojic and Bathe, 2005). These models are built into our code PAK.

For the efficiency and overall applicability of a nonlinear material model, one of the crucial elements is to have a consistent tangent constitutive matrix. As shown in our references, the GPM methodology provides the basis for the derivation of such a matrix. We will show here an example of the effects of the consistent tangent elastic-plastic matrix.

A computational procedure is presented in the next section for a case of the elastic-plastic material model with a general form of yield condition, followed in the next section by the application of the GPM to the Cam-clay geological elastic-plastic model. We summarize in the final section the role of the GPM in the development of the PAK program for the analysis of inelastic material deformation.

## 2. A general form of time-independent plasticity material model

According to (Kojic and Bathe, 2005), in the incremental elastic-plastic analysis of a material body, the unknown quantities at the end of the time step $\Delta t$ are

$$ {}^{t+\Delta t}\boldsymbol{\sigma}, {}^{t+\Delta t}\boldsymbol{\beta}, {}^{t+\Delta t}\mathbf{e}^{IN} \tag{1} $$

where ${}^{t+\Delta t}\boldsymbol{\sigma}, {}^{t+\Delta t}\boldsymbol{\beta}, {}^{t+\Delta t}\mathbf{e}^{IN}$ are stress, internal variables, and inelastic strains. It is assumed that these variables are known at the start of time step, as well as strains $\mathbf{e}^{t+\Delta t}$. The assumption is that there is a governing parameter $p$, such that the th unknown (1) can be expressed as

$$ \begin{aligned} {}^{t+\Delta t}\boldsymbol{\sigma} &= \boldsymbol{\sigma}\left( {}^{t}\boldsymbol{\sigma}, {}^{t}\mathbf{e}, {}^{t}\boldsymbol{\beta}, {}^{t}\mathbf{e}^{IN}, {}^{t+\Delta t}\mathbf{e}, p \right) \\ {}^{t+\Delta t}\boldsymbol{\beta} &= \boldsymbol{\beta}\left( {}^{t}\boldsymbol{\sigma}, {}^{t}\mathbf{e}, {}^{t}\boldsymbol{\beta}, {}^{t}\mathbf{e}^{IN}, {}^{t+\Delta t}\mathbf{e}, p \right) \\ {}^{t+\Delta t}\mathbf{e}^{IN} &= \mathbf{e}^{IN}\left( {}^{t}\boldsymbol{\sigma}, {}^{t}\mathbf{e}, {}^{t}\boldsymbol{\beta}, {}^{t}\mathbf{e}^{IN}, {}^{t+\Delta t}\mathbf{e}, p \right) \end{aligned} \tag{2} $$

where the upper index $t$ denotes values at the start of the time step. It is fundamental that, a monotonic function $f(p)$ can be formulated such that the zero of that function enables determination of all unknowns (1), i.e.,

$$ f(p) = 0 \tag{3} $$

This methodology is illustrated in Fig.1b, with the aim to show an example of the internal variable of a material model; here it is the so-called back stress $\boldsymbol{\alpha}$ which specifies the center of the yield surface in the deviatoric plane. The constitutive relation for this internal variable is

$$ \Delta\boldsymbol{\alpha} = \frac{2}{3}\left( \bar{E}_P - M\hat{E}_P \right)\Delta\mathbf{e}^P \tag{4} $$

where $\bar{E}_P$ and $\hat{E}_P$ are weighted plastic moduli according to the yield curve, $\Delta\mathbf{e}^P$ is the increment of plastic strain, and M is the mixed hardening parameter. The governing parameter is the modulus of plastic strain $\left\|\Delta\mathbf{e}^P\right\|$. Details of the computational procedure are given in (Kojic and Bathe, 2005).

To illustrate the generality of the GPM we here outline the stress integration in the case of a general form of the yield surface (Kojic and Bathe, 2005). Schematic representation of the

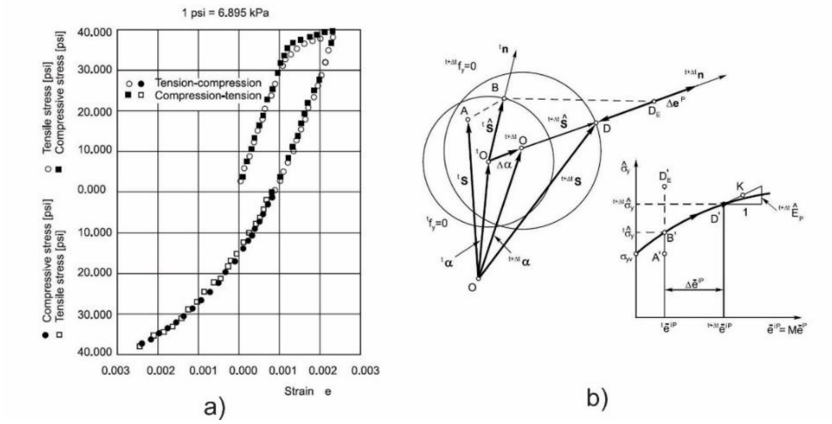


**Fig. 1** a) Hysteresis of metal in loading-reverse loading plastic deformation of steel, according to Smith and Sidebottom (1965) ; b) Graphical representation of the stress integration at a load step. Yield surfaces in the deviatoric plane at the start and end of the load step. Stress points *B* and *D* at yield surfaces, and *B*' and *D*' at the yield curve correspond to the start and end of the load step. The internal variable is back stress $\boldsymbol{\alpha}$. According to (Kojic and Bathe, 2005)

computational procedure is shown in Fig. 2. The basic relations used here include the following., The increment of plastic strain can be expressed as

$$\Delta\mathbf{e}^P = \left\|\Delta\mathbf{e}^P\right\|\mathbf{n} \tag{5}$$

where $\mathbf{n}$ is the unit normal to the yield surface (with double shear terms). Then, the following relations follow from the constitutive relations for stress and internal variables, and the yield condition :

$$f\left(\left\|\Delta\mathbf{e}^P\right\|\right) = {}^{t+\Delta t}f_y\left({}^{t+\Delta t}\boldsymbol{\sigma}^E - \left\|\Delta\mathbf{e}^P\right\|\mathbf{C}^E\,{}^{t+\Delta t}\mathbf{n},\,{}^t\boldsymbol{\beta} - \left\|\Delta\mathbf{e}^P\right\|\mathbf{C}^P_\beta\,{}^{t+\Delta t}\mathbf{n}_\beta\right) \tag{6}$$

where $f_y$ is the yield function, ${}^{t+\Delta t}\boldsymbol{\sigma}^E$ is the stress corresponding to the elastic solution,

$${}^{t+\Delta t}\mathbf{n}_\beta = {}^{t+\Delta t}\left(\frac{\partial f_y}{\partial\boldsymbol{\beta}} / \left|\frac{\partial f_y}{\partial\boldsymbol{\beta}}\right|\right) \tag{7}$$

and $\mathbf{C}^E$ and $\mathbf{C}^P_\beta$ are elastic matrix and constitutive matrix related to the internal variables. We use a trial value $\left\|\Delta\mathbf{e}^P\right\|^{(k)}$ and evaluate the trial value $f_y^{(k)}$ of the function (6) using the trial values for stresses and internal variables:

$$\begin{aligned} {}^{t+\Delta t}\boldsymbol{\sigma}^{(k)} &= {}^{t+\Delta t}\boldsymbol{\sigma}^{(k-1)} - \left(\left\|\Delta\mathbf{e}^P\right\|^{(k)} - \left\|\Delta\mathbf{e}^P\right\|^{(k-1)}\right)\mathbf{C}^E\,{}^{t+\Delta t}\mathbf{n}^{(k-1)} \\ {}^{t+\Delta t}\boldsymbol{\beta}^{(k)} &= {}^{t+\Delta t}\boldsymbol{\beta}^{(k-1)} - \left(\left\|\Delta\mathbf{e}^P\right\|^{(k)} - \left\|\Delta\mathbf{e}^P\right\|^{(k-1)}\right)\mathbf{C}^P_\beta\,{}^{t+\Delta t}\mathbf{n}_\beta^{(k-1)} \end{aligned} \tag{8}$$

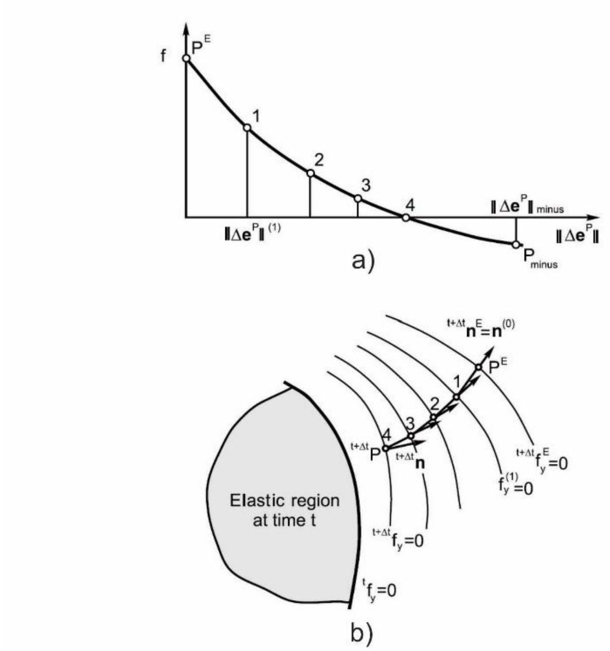


**Fig. 2** Solution procedure for the elastic-plastic model with a general form of the yield condition. a) Search for the zero of the function of the governing parameter $\left\|\Delta \mathbf{e}^P\right\|$; b) Successive yield surfaces analogous to the return mapping concept. According to (Kojic and Bathe, 2005).

The trials continue until the zero of the function (6) is obtained. Additional details are given in (Kojic and Bathe, 2005). The return mapping was introduced in (Simo and Taylor, 1986) , practically at the same time as we published our 'effective-stress-function' (Kojic and Bathe, 1987).

It is very important to emphasize the role of the consistent-tangent elastic-plastic matrix. The general form of this matrix can be written as

$$ {}^{t+\Delta t}C_{ijrs} = \left.\frac{\partial\, {}^{t+\Delta t}\sigma_{ij}}{\partial\, {}^{t+\Delta t}e_{rs}}\right|_{p=\mathrm{const}} + \frac{\partial\, {}^{t+\Delta t}\sigma_{ij}}{\partial\, {}^{t+\Delta t}p}\frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}e_{rs}} \tag{9} $$

We do not present additional details of the derivation of this matrix for specific material models – they are given in our references. Here is given illustration of the effects of the tangent character of this matrix. Fig. 3 displays convergence to the exact solution in the case of torsion of a tube where elastic-plastic deformation is produced. The material is metal with a bilinear yield curve. It can be seen that the number of iterations is 346 is needed if the elastic matrix is used (modified Newton method), while the solution is obtained by 2 iterations only if the consistent elastic-plastic matrix is employed (full Newton method). The consistent matrices are built into the program PAK.

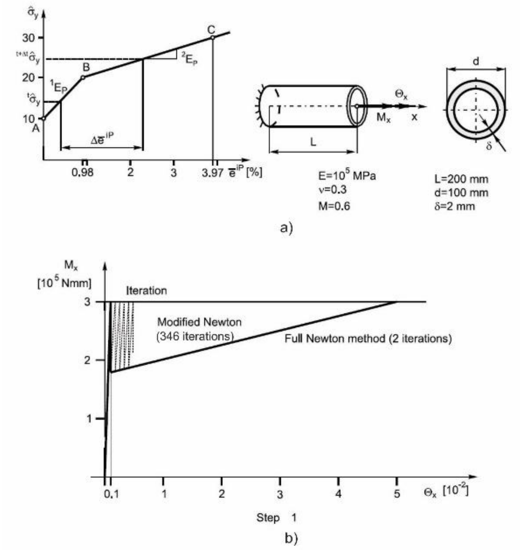


**Fig. 3** a) Torsion of a tube, elastic-plastic deformation, von Mises yield condition; b)Solution is reached after 346 iterations if the elastic matrix is used (modified Newton method), while only 2 iterations are needed when the consistent elastic-plastic matrix is employed (full Newton method). According to (Kojic and Bathe, 2005).

Accuracy of the ESF is demonstrated on numerous examples in the book (Kojic and Bathe, 2005) and during the Kojic implementation of the ESF into the ADINA code, and later into the PAK. In Fig. 4, we show the field of effective plastic deformation in the supporting thin-walled beam with a closed cross-section used for bridges, subjected to torsion, which produces elastic-plastic deformation. The PAK FE model was generated at the Laboratory for Engineering Software of the Mechanical Engineering Faculty in Kragujevac, and the experiment was carried out at the Civil Engineering Faculty in Belgrade (Ph. D. thesis of Bratislav Stipanic under the mentorship of academican Nikola Hajdin). Stipanic was a professor and the main collaborator in designing bridges over the largest rivers in Europe: the Danube, the Sava in Serbia, and the Visla in Poland. The computational model results and experimental measurements were perfectly matched.

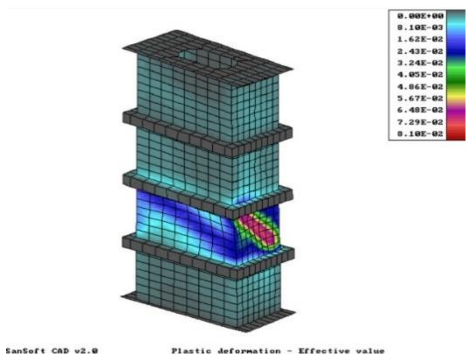


**Fig. 4.** Field of plastic deformation within a thin-walled beam with closed cross-section subjected to torsion. This beam structure is used to carry the load at bridges (Ph. D. thesis of Bratislav Stipanic, Mentor academician Nikola Hajdin, 1989)

## 3. Application of the GPM to the Cam-clay geological material model

In this section, we apply our GPM concept to the Cam-clay geological material model. We briefly summarize a part of the computational procedure given in (Kojic and Bathe, 2005) as a simple example built into our PAK code.

The model is represented by the yield surface shown in Fig. 4. The yield function, according to Wood (1990) in the plane second invariant of the deviatoric stress $J_{2D}$ – mean stress $\sigma_m$ , is shown in Fig. 5. The analytical form is

$$f_y = \sigma_m\left(\sigma_m - p_0\right) + \frac{3J_{2D}}{M^2} = 0 \tag{10}$$

where $p_0$ is the length of the horizontal axis of the ellipse, and M is a material parameter. In the computational procedure, we distinguish three loading regimes, comparing the elastic mean stress ${}^{t+\Delta t}\sigma_m^E$ and ${}^{t}p_0$ :

$$\begin{aligned} &\text{hardening} && {}^{t+\Delta t}\sigma_m^E > \frac{1}{2}\,{}^{t}p_0 \\ &\text{softening} && {}^{t+\Delta t}\sigma_m^E < \frac{1}{2}\,{}^{t}p_0 \\ &\text{perfect plasticity} && {}^{t+\Delta t}\sigma_m^E = \frac{1}{2}\,{}^{t}p_0 \end{aligned} \tag{11}$$

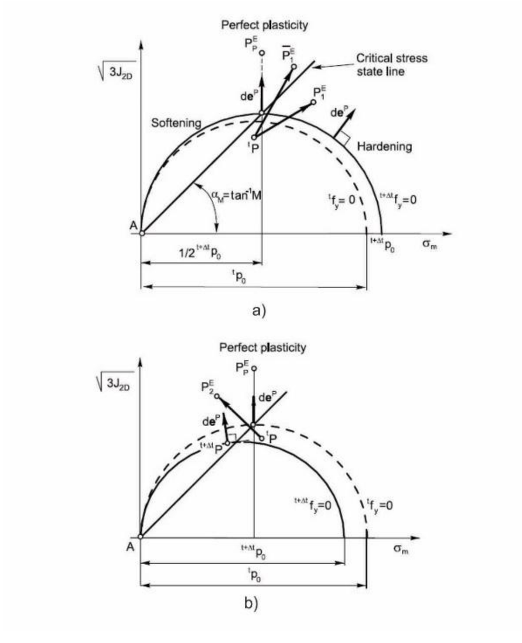


**Fig. 5.** Cam-clay model. Loading regimes at a time step. a) Hardening and perfect plasticity; b) Softening. According to (Kojic and Bathe, 2005).

We select the increment of the mean volumetric deformation $\Delta e_m^P$ as the governing parameter and then, for a selected value of $\Delta e_m^P$ we evaluate

- void ratio $^{t+\Delta t}e = \left(1+{}^0e\right)\exp\left(-{}^{t+\Delta t}e_V\right)-1$
- quantity $^{t+\Delta t}b_V = \dfrac{k_s}{3\left(1+{}^{t+\Delta t}e\right)}$
- $^{t+\Delta t}p_0 = {}^t p_0 \exp\left(\dfrac{\Delta e_m^P}{^{t+\Delta t}b_V}\right)$
- $^{t+\Delta t}\sigma_m = c_m\left({}^{t+\Delta t}e_m - {}^t e_m - \Delta e_m^P\right)$
- $\Delta\lambda = \dfrac{3\Delta e_m^P}{2\,{}^{t+\Delta t}\sigma_m - {}^{t+\Delta t}p_0}$
- $^{t+\Delta t}\mathbf{S} = \dfrac{2G\left({}^{t+\Delta t}\mathbf{e}' - {}^t\mathbf{e}'^P\right)}{1+6\Delta\lambda G/M^2}, \quad {}^{t+\Delta t}J_{2D} = \dfrac{1}{2}\,{}^{t+\Delta t}\mathbf{S}\bullet{}^{t+\Delta t}\mathbf{S}$
- Check if $\left|{}^{t+\Delta t}f_y\right| \le 0$

where $^{t+\Delta t}e_V$ is volumetric strain; $^{t+\Delta t}\mathbf{e}'$ and $^t\mathbf{e}'^P$ are deviatoric total and plastic strains; $c_m$=3$K$ is volumetric elastic constant ($K$ is volumetric modulus); $G$ is the shear modulus; $k_s$ is a material parameter. If the convergence is not reached, use another trial value of the governing parameter $\Delta e_m^P$.

In the case of perfect plasticity, we have that the increment of the mean volumetric strain is equal to zero, i.e. $\Delta e_m^P = 0$, $^{t+\Delta t}\sigma_m = {}^{t+\Delta t}\sigma_m^E = 0.5\,{}^t p_0$ and $\Delta\lambda$ is

$$\Delta\lambda = \frac{M^2}{6G}\left(\frac{\sqrt{3\,{}^{t+\Delta t}J_{2D}^E}}{M\,{}^{t+\Delta t}\sigma_m^E} - 1\right) \tag{12}$$

with $\Delta\lambda$ determined, we continue as in the other cases given above. The derivation for the elastic-plastic matrix is given in our reference. The application of this model is illustrated by a model of the triaxial compression test, used in a standard evaluation of characteristics of a geological material. The data and the results are shown in Fig. 6, which agree with those reported by Desai and Siriwardane (1984).

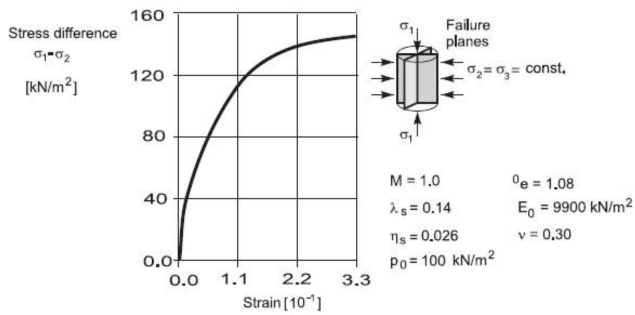


**Fig. 6** Triaxial compression test of Cam-clay material. Model data and results that agree with those of Desai and Siriwardane (1984). According to (Kojic and Bathe, 2005).

## 4. Concluding remarks

We have outlined our governing parameter method (GPM), generalized from the ESF, as the basic concept for the implicit stress integration of inelastic material deformation. This methodology is built within our PAK finite element program, developed over decades. The GPM with the implicit stress integration has been accepted by numerous developers and users of our code as fundamental for computational procedures for inelastic material models within the PAK package. This outline includes a general elastic-plastic model and the Cam-clay model as a brief insight into our methodology, while a detailed description of it is available in our numerous references.

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