Journal of the Serbian Society for Computational Mechanics / Vol. 9 / No. 1, 2015 / pp. 15-35

(UDC: 624.072.2.042.1)

Stress analysis of laminated composite and soft core sandwich beams using a simple higher order shear deformation theory

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Abstract

In this paper, the refined beam theory (RBT) is examined for the bending of simply supported isotropic, laminated composite and sandwich beams. The axial displacement field uses parabolic function in terms of thickness ordinate to include the effect of transverse shear deformation. The transverse displacement consists of bending and shear components. The present theory satisfies the traction free conditions on the upper and lower surfaces of the beam without using problem dependent shear correction factors of Timoshenko. Governing differential equations and boundary conditions associated with the assumed displacement field are obtained by using the principle of virtual work. To prove the credibility of the present theory, we applied it to the bending analysis of beams. A simply supported isotropic, laminated composite and sandwich beams are analyzed using Navier approach. The numerical results of non-dimensional displacements and stresses obtained by using the present theory are presented and compared with those of other refined theories available in the literature along with the elasticity solution.

Keywords: transverse shear deformation, shear correction factor, transverse shear stress, bending, laminated composite, sandwich.

1. Introduction

Structural components made of fibrous composite materials are increasingly being used in various engineering applications due to their attractive properties in strength, stiffness, and lightness. The effect of transverse shear deformation is more pronounced in thick beams made of fibrous composite material which has a high extensional modulus to shear modulus ratio.

The classical beam theory (CBT) does not predict the correct bending behaviour of thick beams made of fibrous composite materials. The first order shear deformation beam theory (FSDT) developed by Timoshenko (1921) includes the effect of transverse shear deformation but does not satisfy the zero shear stress conditions on the top and bottom surfaces of the beam, hence, it requires shear correction factor. Many higher order theories are available in the literature for the bending, buckling and free vibration analysis of laminated composite beams which take into account the effect of transverse shear deformation and do not require shear correction factor. The third order theory of Reddy (1984) is the most commonly used higher order theory for beams as well as for plates. A recent review of higher order theories available for the analysis of laminated composite beams has been presented by Ghugal and Shimpi (2001). Kadoli et al. (2008) applied the third order theory of Reddy for the static analysis of functionally graded beams. A general analytical model was developed by Lee (2005) using the shear deformable beam theory and was applied to the flexural analysis of thin walled I-shaped laminated composite beams. Chen and Wu (2005) developed a new higher-order shear deformation theory based on global-local superposition technique. Reddy (2007) reformulated various beam theories using nonlocal elasticity and applied them to the bending, buckling and vibration analysis of beams. Wang et al. (2008) also presented some work on beam bending solutions based on nonlocal Timoshenko beam theory. Mechab et al. (2008) carried out an assessment of parabolic and exponential shear deformation theories on bending of short laminated composite beams subjected to mechanical and thermal loadings. Carrera and Giunta (2010) presented refined beam theories based on a unified formulation and applied them to the static analysis of beams made of isotropic materials. Karama et al. (2008) did the refinement of Ambartsumian multi-layer beam theory considering an exponential function in terms of thickness coordinate. Chakrabarti et al. (2011) presented a new finite element model based on the zig-zag theory for the analysis of sandwich beams which is further extended by Chalak et al. (2011) for free vibration analysis of laminated sandwich beams having soft core. Gherlone et al. (2011) carried out the finite element analysis of multilayered composite and sandwich beams based on the refined zigzag theory. Sayyad and Ghugal (2011) developed a trigonometric shear and normal deformation theory for the bending analysis of laminated composite beams subjected to various static loadings. Sayyad (2011) presented a refined shear deformation theory for the static flexure and free vibration analysis of thick isotropic beams considering parabolic, trigonometric, hyperbolic and exponential functions in terms of thickness co-ordinate associated with transverse shear deformation effect. This theory is further extended by Sayyad et al. (2014) for the flexural analysis of single layered composite beams. Chen et al. (2011) carried out bending analysis of laminated composite plates considering first order shear deformation based on modified couple stress theory. Aguiar et al. (2012) carried out static analysis of composite beams of different cross-sections using mixed and displacement based models. Ghugal and Shinde (2013) extended the layerwise trigonometric shear deformation theory of Shimpi and Ghugal (2001) for the bending analysis of two layered anti-symmetric laminated composite beams with various boundary conditions. Recently, Sayyad et al. (2015) developed a new trigonometric shear deformation theory for the bending analysis of laminated composite and sandwich beams.

The theory used in the present study is originally developed by Shimpi and Patel (2006) for the bending analysis of orthotropic plates. In this paper, this theory is applied to the bending analysis of laminated composite and sandwich beams. Governing equations and boundary conditions of the presented theory are obtained using the principle of virtual work. The Navier's solution technique is employed for the simply supported boundary conditions. The numerical results are obtained for isotropic, laminated composite and sandwich beams subjected to sinusoidal load.

2. The development of the theory

A laminated composite beam of length 'L', width 'b' and overall thickness 'h' as shown in Fig. 1 is considered. The beam consists of 'N' number of layers made up of linearly elastic orthotropic material. The beam occupies the region $0 \le x \le L$, $-b/2 \le y \le b/2$ and $-h/2 \le z \le h/2$ in Cartesian coordinate system.

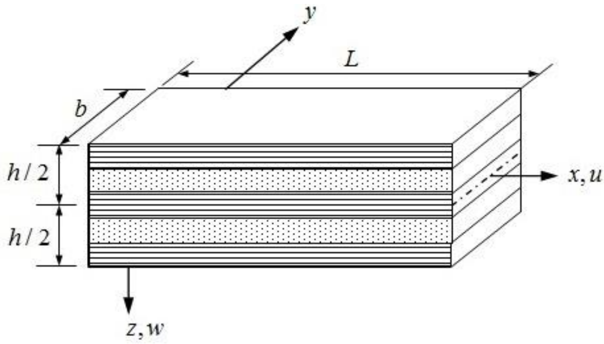

Fig. 1. Geometry and coordinate system of laminated composite beam.

In the present theory, the axial displacement u in x direction consists of extension, bending and shear components, whereas transverse displacement w in the z-direction consists of bending (w_b) and shear (w_s) components along the center line of the beam:

$$u(x,z) = u_0(x) - z\frac{dw_b(x)}{dx} - \left[\frac{5}{3}\frac{z^3}{h^2} - \frac{z}{4}\right]\frac{dw_s(x)}{dx} \tag{1}$$

$$w = w_b(x) + w_s(x) \tag{2}$$

where u_0 is the axial displacement along the center line of the beam. The nonzero strain components corresponding to the assumed displacement field are as follows:

$$\varepsilon_x = \varepsilon_x^0 + z k_x^b + f(z) k_x^s \quad \text{and} \quad \gamma_{zx} = \gamma_{zx}^0\, g(z) \tag{3}$$

where

$$\varepsilon_x^0 = \frac{du_0}{dx}, \quad k_x^b = -\frac{d^2 w_b}{dx^2}, \quad k_x^s = -\frac{d^2 w_s}{dx^2}, \quad \gamma_{zx}^0 = \frac{dw_s}{dx}, \quad f(z) = \left[\frac{5}{3}\frac{z^3}{h^2} - \frac{z}{4}\right] \quad \text{and} \quad g(z) = \left[\frac{5}{4} - 5\frac{z^2}{h^2}\right] \tag{4}$$

The stress strain relationship for k^{th} layer of laminated composite beam is as follows:

$$\sigma_x^k = Q_{11}^k\, \varepsilon_x^k \quad \text{and} \quad \tau_{zx}^k = Q_{55}^k\, \gamma_{xz}^k \tag{5}$$

where Q_{11}^k is the Young's modulus in the axial direction of the laminated composite beam, while Q_{55}^k is the shear modulus. The principle of virtual work is used to obtain the governing equations of equilibrium and associate boundary conditions. The analytical form of the Principle of virtual work is:

$$b\int_0^L \int_{-h/2}^{h/2} \left(\sigma_x \delta\varepsilon_x + \tau_{zx}\delta\gamma_{xz}\right) dx\, dz - b\int_0^L q\left(\delta w_b + \delta w_s\right) dx = 0 \tag{6}$$

Substituting expressions for strains and stresses from Eqs. (3) - (5) into Eq. (6), the principle of virtual work can be rewritten as:

$$\int_0^L \left(N_x \frac{d\delta u_0}{dx} - M_x^b \frac{d^2\delta w_b}{dx^2} - M_x^s \frac{d^2\delta w_s}{dx^2} + Q_x \frac{d\delta w_s}{dx} \right) dx - \int_0^L q\left(\delta w_b + \delta w_s\right) dx = 0 \tag{7}$$

where δ is the variational operator. The stress resultants $\left(N_x, M_x^b, M_x^s, Q_x\right)$ associated with the assumed displacement field are defined as:

$$\left\{N_x \quad M_x^b \quad M_x^s\right\} = \sum_{k=1}^{N} \int_{-h/2}^{h/2} \left\{1 \quad z \quad f(z)\right\} \sigma_x^k \, dz, \quad Q_x = \sum_{k=1}^{N} \int_{-h/2}^{h/2} \tau_{xz}^k \, g(z) dz \tag{8}$$

Substituting stresses from Eq. (5) into the Eq. (8) and integrating through the thickness, the following equations are obtained:

$$N_x = A_{11} \frac{du_0}{dx} - B_{11} \frac{d^2 w_b}{dx^2} - C_{11} \frac{d^2 w_s}{dx^2} \tag{9}$$

$$M_x^b = B_{11} \frac{du_0}{dx} - D_{11} \frac{d^2 w_b}{dx^2} - E_{11} \frac{d^2 w_s}{dx^2} \tag{10}$$

$$M_x^s = C_{11} \frac{du_0}{dx} - E_{11} \frac{d^2 w_b}{dx^2} - F_{11} \frac{d^2 w_s}{dx^2} \tag{11}$$

$$Q_x = G_{55} \frac{dw_s}{dx} \tag{12}$$

Integrating Eq. (7) by parts and setting the coefficients of $\delta u_0, \delta w_b, \delta w_s$ zero, the following governing differential equations and associated boundary conditions are obtained:

$$\begin{aligned} \delta u_0 &: \quad \frac{dN_x}{dx} = 0 \\ \delta w_b &: \quad \frac{d^2 M_x^b}{dx^2} + q = 0 \\ \delta w_s &: \quad \frac{d^2 M_x^s}{dx^2} + \frac{dQ_x}{dx} + q = 0 \end{aligned} \tag{13}$$

The boundary conditions of the present theory at *x* = 0, *x* = *L* are of the form:

$$\begin{aligned} &\text{Specify} \quad N_x \quad \text{or} \quad u_0 \\ &\text{Specify} \quad \frac{dM_x^b}{dx} \quad \text{or} \quad w_b \\ &\text{Specify} \quad M_x^b \quad \text{or} \quad \frac{dw_b}{dx} \\ &\text{Specify} \quad \frac{dM_x^s}{dx} \quad \text{or} \quad w_s \\ &\text{Specify} \quad M_x^s \quad \text{or} \quad \frac{dw_s}{dx} \end{aligned} \tag{14}$$

The governing differential equations in terms of unknown displacement variables (u_0, w_b, w_s) are rewritten as:

$$\delta u_0: \quad -A_{11}\frac{d^2u_0}{dx^2}+B_{11}\frac{d^3w_b}{dx^3}+C_{11}\frac{d^3w_s}{dx^3}=0 \tag{15}$$

$$\delta w_b: \quad -B_{11}\frac{d^3u_0}{dx^3}+D_{11}\frac{d^4w_b}{dx^4}+E_{11}\frac{d^4w_s}{dx^4}=q \tag{16}$$

$$\delta w_s: \quad -C_{11}\frac{d^3u_0}{dx^3}+E_{11}\frac{d^4w_b}{dx^4}+F_{11}\frac{d^4w_s}{dx^4}-G_{55}\frac{d^2w_s}{dx^2}=q \tag{17}$$

where

$$\begin{gathered} A_{11}=\sum_{k=1}^{N}Q_{11}^{k}\int_{-h/2}^{h/2}dz,\quad B_{11}=\sum_{k=1}^{N}Q_{11}^{k}\int_{-h/2}^{h/2}z\,dz,\quad C_{11}=\sum_{k=1}^{N}Q_{11}^{k}\int_{-h/2}^{h/2}f\left(z\right)dz,\\ D_{11}=\sum_{k=1}^{N}Q_{11}^{k}\int_{-h/2}^{h/2}z^2\,dz,\quad E_{11}=\sum_{k=1}^{N}Q_{11}^{k}\int_{-h/2}^{h/2}z\,f\left(z\right)dz,\\ F_{11}=\sum_{k=1}^{N}Q_{11}^{k}\int_{-h/2}^{h/2}\left[f\left(z\right)\right]^2dz,\quad G_{55}=\sum_{k=1}^{N}Q_{55}^{k}\int_{-h/2}^{h/2}\left[g\left(z\right)\right]^2dz \end{gathered} \tag{18}$$

2.1 The Navier solution for simply supported beams

The closed form solution is obtained using the Navier's solution technique. A beam as shown in Fig. 1 is considered for the detailed numerical study. The following simply-supported boundary conditions are considered at x = 0, x = L

$$N_x=w_b=w_s=M_x^{\,b}=M_x^s=0 \tag{19}$$

The beam is subjected to sinusoidal load $q(x)$ on the top surface, *i.e.* z = -h/2. The load q is expanded in single trigonometric series:

$$q\left(x\right)=q_0\sin\frac{\pi x}{L} \tag{20}$$

where q_0 denotes the intensity of the load at the center of the beam. The following expansions of the unknown displacement variables (u_0, w_b, w_s) satisfy the boundary conditions in Eq. (19):

$$u_0=u_1\cos\frac{\pi x}{L},\quad w_b=w_{b1}\sin\frac{\pi x}{L},\quad w_s=w_{s1}\sin\frac{\pi x}{L} \tag{21}$$

where u_1, w_{b1}, w_{s1} are arbitrary parameters. Substituting unknown displacement variables (u_0, w_b, w_s) from Eq. (21) and the load from Eq. (20) into the Eqs. (15) - (17), the closed-form solutions can be obtained from the following equations:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13}\\ K_{12} & K_{22} & K_{23}\\ K_{13} & K_{23} & K_{33}\end{bmatrix}\begin{Bmatrix} u_1\\ w_{b1}\\ w_{s1}\end{Bmatrix}=\begin{Bmatrix} 0\\ q_0\\ q_0\end{Bmatrix} \tag{22}$$

where

$$K_{11}=A_{11}\frac{\pi^2}{L^2},\quad K_{12}=-B_{11}\frac{\pi^3}{L^3},\quad K_{13}=-C_{11}\frac{\pi^3}{L^3},$$
$$K_{22}=D_{11}\frac{\pi^4}{L^4},\quad K_{23}=E_{11}\frac{\pi^4}{L^4},\quad K_{33}=F_{11}\frac{\pi^4}{L^4}+G_{55}\frac{\pi^2}{L^2} \tag{23}$$

3. Numerical results and discussion

To assess the efficiency of the present theory, the bending analysis of simply supported beams is considered. The numerical results are obtained for displacements and stresses for isotropic, laminated composite and sandwich beams. The values of transverse shear stress (τ_{zx}) presented in the tables are obtained by using equilibrium equations of the theory of elasticity to satisfy interface continuity.

$$\tau_{zx}=\sum_{k=1}^{N}\int_{-h/2}^{h/2}\frac{d\sigma_x}{dx}dz \tag{24}$$

The following non-dimensional forms are used to present the displacements and stresses:

$$\bar{u}(0,-h/2)=u\times n_1,\quad \bar{w}(L/2,0)=w\times n_2,$$
$$\bar{\sigma}_x(0,-h/2)=\sigma_x\times n_3,\quad \bar{\tau}_{zx}(0,0)=\tau_{zx}\times n_3 \tag{25}$$

where

$$n_1=\frac{b}{100\,q_0 h},\quad n_2=\frac{100\,h^3}{q_0 a^4},\quad n_3=\frac{b}{10\,q_0} \tag{26}$$

3.1 Bending analysis of isotropic beams

The simply supported isotropic beams subjected to sinusoidal load are considered with the following material properties:

$$Q_{11}=210\,\text{GPa}\quad\text{and}\quad Q_{55}=80.77\,\text{GPa}$$

Table 1 shows the maximum displacements and stresses for the simply supported isotropic beams with *L*/*h* = 4, 10, 20, 50 and 100. The present results are compared with the elasticity solution provided by Ghugal (2006), the higher order shear deformation theory (HSDT) of Reddy (1984), the first order shear deformation theory (FSDT) of Timoshenko (1921) and the classical beam theory (CBT). From the examination of Table 1 it is observed that the present theory accurately predicts the values of axial ($\bar{u}$) and transverse ($\bar{w}$) displacements. For *L*/*h* = 4, 10 and 20, these displacements ($\bar{u}$ and $\bar{w}$) are identical to those obtained by the HSDT of Reddy (1984). The bending stress predicted by the present theory is in excellent agreement with that of the exact solution. It is also observed that the transverse shear stress ($\bar{\tau}_{zx}$) evaluated by using equilibrium equations is close to elasticity solution.

3.2 Bending analysis of two layered ($0^0/90^0$) laminated composite beams

The two layered anti-symmetric cross-ply laminated composite beams with simply supported boundary conditions and subjected to sinusoidal load with following material properties are considered.

0^0 layer (z = -h/2 to z = 0): Q_{11} = 25 and Q_{55} = 0.5

90^0 layer (z = 0 to z = h/2): Q_{11} = 1.0 and Q_{55} = 0.2

The layers are of equal thickness *i.e.* h/2. The displacements and stresses are obtained for different L/h ratios such as 4, 10, 20, 50 and 100. The numerical results are reported in Table 2. From Table 2 it is observed that, even for thick beams, the displacements and stresses obtained using the present theory are in excellent agreement with those obtained by the HSDT of Reddy (1984) and the 3-D elasticity solution given by Pagano (1969). The transverse shear stress continuity is maintained *via* equilibrium equations of theory of elasticity. The CBT underestimates the values of displacements and bending stress whereas it overestimates the values of transverse shear stress due to the neglect of transverse shear deformation. The variations of axial displacement, bending stress and transverse shear stress with respect to thickness ordinate are shown in Fig. 2 through Fig. 4.

3.3 Bending analysis of three layered ($0^0/90^0/0^0$) laminated composite beams

A simply supported three layered symmetric cross-ply laminated composite beam under sinusoidal load is considered with the following material properties.

0^0 layer (z = -h/2 to z = -h/6): Q_{11} = 25 and Q_{55} = 0.5

90^0 layer (z = -h/6 to z = h/6): Q_{11} = 1.0 and Q_{55} = 0.2

0^0 layer (z = h/6 to z = h/2): Q_{11} = 25 and Q_{55} = 0.5

The layers are of equal thickness i.e. h/3. The displacements and stresses for the beam with above material properties are presented in Table 3. The numerical results are compared with the HSDT of Reddy (1984), the FSDT of Timoshenko (1921), the CBT and exact elasticity solution given by Pagano (1969). Comparing the results with other theories, it is observed that, axial displacement predicted by the present theory and the theory of Reddy is identical for all L/h ratios whereas maximum transverse displacement is in excellent agreement with that of the exact solution. The through thickness distribution of axial displacement (L/h = 4) is plotted in Fig. 5. The FSDT and CBT underestimate the values of bending stress whereas they overestimate the transverse shear stress compared to those of the exact solution. It is also pointed out that, the bending and transverse stresses obtained using the FSDT and CBT are identical. The present theory and the theory of Reddy show excellent agreement for these stresses. The through thickness distributions of these stresses ($\bar{\sigma}_x$, $\bar{\tau}_{zx}$) are shown in Fig. 6 and Fig.7.

3.4 Bending analysis of simply supported three layered (0^0/core/0^0) sandwich beams

A three layered simply supported soft sandwich beam under sinusoidal load is analyzed using the following properties:

0^0 layer (z = -0.5h to z = -0.4h): Q_{11} = 25 and Q_{55} = 0.5

core (z = -0.4h to z = 0.4h): Q_{11} = 4.0 and Q_{55} = 0.06

0^0 layer (z = 0.4h to z = 0.5h): Q_{11} = 25 and Q_{55} = 0.5

The thickness of each face sheet is 0.1h and core is of 0.8h. The maximum displacements and stresses for L/h = 4, 10, 20, 50 and 100 are given in Table 4. The exact elasticity solution for this problem is not available in the literature, therefore, the results are also generated by using the HSDT, FSDT and CBT. From Table 4 it is observed that the present theory is in excellent agreement with the HSDT of Reddy (1984) while predicting displacements and bending stress, but it predicts the lower value of transverse shear stress. The FSDT and CBT show identical values for axial displacement and bending stress for all L/h ratios. The through thickness distributions of axial displacement, bending stress and transverse shear stress are plotted in Fig. 8 through Fig. 10.

3.5 Bending analysis of simply supported five layered ($0^0/90^0$/core/$90^0/0^0$) sandwich beams

A simply supported five layered symmetric soft sandwich beam under sinusoidal load is considered with the following material properties:

0^0 layer (z = -0.5h to z = -0.45h): Q_{11} = 25 and Q_{55} = 0.5

90^0 layer (z = -0.45h to z = -0.4h): Q_{11} = 1.0 and Q_{55} = 0.2

core (z = -0.4h to z = 0.4h): Q_{11} = 4.0 and Q_{55} = 0.06

90^0 layer (z = 0.4h to z = 0.45h): Q_{11} = 1.0 and Q_{55} = 0.2

0^0 layer (z = 0.45h to z = 0.5h): Q_{11} = 25 and Q_{55} = 0.5

The thickness of each face sheet is 0.05h and core is of 0.8h. The displacements and stresses obtained for different L/h ratios are reported in Table 2. From this table, it is noted that the displacements and stresses evaluations using the present theory match with the HSDT of Reddy (1984) whereas the FSDT of Timoshenko (1921) and the CBT underestimate the displacements and bending stress. Fig. 11 through Fig. 13 shows through thickness distributions of axial displacement, bending stress and transverse shear stress for this loading case.

L/h	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
4	Present	RBT	0.1271	1.429	0.9986	0.1893
	Reddy (1984)	HSDT	0.1271	1.429	0.9986	0.1897
	Timoshenko (1921)	FSDT	0.1238	1.430	0.9727	0.1910
	Bernoulli-Euler	CBT	0.1238	1.232	0.9727	0.1910
	Ghugal (2006)	Exact	0.1230	1.411	0.9958	0.1900
10	Present	RBT	1.9434	1.264	6.1052	0.4767
	Reddy (1984)	HSDT	1.9434	1.264	6.1050	0.4769
	Timoshenko (1921)	FSDT	1.9351	1.264	6.0790	0.4774
	Bernoulli-Euler	CBT	1.9351	1.232	6.0790	0.4774
	Ghugal (2006)	Exact	1.9295	1.261	6.0910	0.4764
20	Present	RBT	15.497	1.2398	24.343	0.9545
	Reddy (1984)	HSDT	15.497	1.2398	24.343	0.9546
	Timoshenko (1921)	FSDT	15.481	1.2398	24.317	0.9549
	Bernoulli-Euler	CBT	15.481	1.2322	24.317	0.9549
	Ghugal (2006)	Exact	---	1.2318	24.194	0.9474
50	Present	RBT	241.927	1.2331	152.007	2.3871
	Reddy (1984)	HSDT	241.916	1.2331	152.004	2.3871
	Timoshenko (1921)	FSDT	241.879	1.2331	151.977	2.3872
	Bernoulli-Euler	CBT	241.886	1.2322	151.981	2.3872
100	Present	RBT	1935.17	1.2322	607.953	4.7744
	Reddy (1984)	HSDT	1935.79	1.2322	608.146	4.7761
	Timoshenko (1921)	FSDT	1935.03	1.2322	607.909	4.7745
	Bernoulli-Euler	CBT	1935.09	1.2322	607.927	4.7745

Table 1. Comparison of displacements stresses for isotropic beam subjected sinusoidal load.

L/h	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
4	Present	RBT	0.0171	4.4514	3.3593	0.2976
	Reddy (1984)	HSDT	0.0171	4.4511	3.3592	0.2883
	Timoshenko (1921)	FSDT	0.0142	4.7966	2.7905	0.2912
	Bernoulli-Euler	CBT	0.0142	2.6254	2.7905	0.2947
	Pagano (1969)	Elasticity	0.0153	4.7080	3.0019	0.2721
10	Present	RBT	0.2294	2.9225	18.019	0.7339
	Reddy (1984)	HSDT	0.2294	2.9225	18.018	0.7263
	Timoshenko (1921)	FSDT	0.2220	2.9728	17.440	0.7279
	Bernoulli-Euler	CBT	0.2220	2.6254	17.440	0.7367
	Pagano (1969)	Elasticity	0.2248	2.9611	17.653	0.7267
20	Present	RBT	1.7912	2.6999	70.342	1.4685
	Reddy (1984)	HSDT	1.7912	2.6999	70.342	1.4550
	Timoshenko (1921)	FSDT	1.7765	2.6978	69.762	1.4558
	Bernoulli-Euler	CBT	1.7765	2.6254	69.762	1.4558
	Pagano (1969)	Elasticity	1.7818	2.7094	69.973	1.4696
50	Present	RBT	27.794	2.6373	436.593	3.6694
	Reddy (1984)	HSDT	27.794	2.6373	436.593	3.6393
	Timoshenko (1921)	FSDT	27.757	2.6370	436.013	3.9396
	Bernoulli-Euler	CBT	27.757	2.6254	436.013	3.6397
	Pagano (1969)	Elasticity	27.766	2.6384	436.150	3.6849
100	Present	RBT	222.133	2.6284	1744.63	7.3382
	Reddy (1984)	HSDT	222.133	2.6284	1744.63	7.2792
	Timoshenko (1921)	FSDT	222.060	2.6283	1744.05	7.2793
	Bernoulli-Euler	CBT	222.059	2.6254	1744.05	7.2798
	Pagano (1969)	Elasticity	222.750	2.6366	1749.50	7.3963

Table 2. Comparison of displacements stresses for two layered ($0^0/90^0$) laminated composite beam subjected sinusoidal load.

L/h	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
4	Present	RBT	0.0086	2.6906	1.6934	0.1648
	Reddy (1984)	HSDT	0.0086	2.7000	1.6989	0.1557
	Timoshenko (1921)	FSDT	0.0051	2.4107	1.0085	0.1769
	Bernoulli-Euler	CBT	0.0051	0.5109	1.0085	0.1769
	Pagano (1969)	Elasticity	0.0092	3.0344	1.8820	0.1430
10	Present	RBT	0.0893	0.8744	7.0171	0.4353
	Reddy (1984)	HSDT	0.0893	0.8751	7.0212	0.4334
	Timoshenko (1921)	FSDT	0.0802	0.8149	6.3033	0.4422
	Bernoulli-Euler	CBT	0.0802	0.5109	6.3033	0.4422
	Pagano (1969)	Elasticity	0.0934	0.9357	7.6660	0.4230
20	Present	RBT	0.6604	0.6023	25.930	0.8797
	Reddy (1984)	HSDT	0.6604	0.6025	25.935	0.8800
	Timoshenko (1921)	FSDT	0.6420	0.5743	25.213	0.8845
	Bernoulli-Euler	CBT	0.6420	0.5109	25.213	0.8801
	Pagano (1969)	Elasticity	0.6695	0.6186	26.320	0.8740
50	Present	RBT	10.078	0.5256	158.298	2.2084
	Reddy (1984)	HSDT	10.078	0.5256	158.308	2.2095
	Timoshenko (1921)	FSDT	10.032	0.5211	157.584	2.2113
	Bernoulli-Euler	CBT	10.032	0.5109	157.584	2.2002
	Pagano (1969)	Elasticity	10.100	0.5283	158.700	2.2050
100	Present	RBT	80.349	0.5146	631.034	4.4194
	Reddy (1984)	HSDT	80.349	0.5146	631.062	4.4217
	Timoshenko (1921)	FSDT	80.257	0.5135	630.339	4.4226
	Bernoulli-Euler	CBT	80.257	0.5109	630.339	4.4004
	Pagano (1969)	Elasticity	80.400	0.5153	631.500	4.4150

Table 3. Comparison of displacements stresses for three layered ($0^0/90^0/0^0$) laminated composite beam subjected sinusoidal load.

L/h	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
4	Present	RBT	0.0183	9.8710	3.5920	0.1530
	Reddy (1984)	HSDT	0.0183	9.8800	3.5920	0.1537
	Timoshenko (1921)	FSDT	0.0087	5.1434	1.7067	0.1549
	Bernoulli-Euler	CBT	0.0087	0.8646	1.7067	0.1549
10	Present	RBT	0.1619	2.4086	12.714	0.3302
	Reddy (1984)	HSDT	0.1615	2.4079	12.684	0.3707
	Timoshenko (1921)	FSDT	0.1358	1.5492	10.666	0.3874
	Bernoulli-Euler	CBT	0.1358	0.8646	10.666	0.3874
20	Present	RBT	1.1421	1.2568	44.848	0.6514
	Reddy (1984)	HSDT	1.1384	1.2543	44.705	0.7663
	Timoshenko (1921)	FSDT	1.0865	1.0358	42.667	0.7748
	Bernoulli-Euler	CBT	1.0865	0.8646	42.667	0.7748
50	Present	RBT	17.166	0.9301	269.652	1.6222
	Reddy (1984)	HSDT	17.106	0.9272	268.715	1.9333
	Timoshenko (1921)	FSDT	16.977	0.8920	266.673	1.9369
	Bernoulli-Euler	CBT	16.977	0.8646	266.673	1.9369
100	Present	RBT	136.55	0.8833	1072.50	3.2425
	Reddy (1984)	HSDT	136.07	0.8803	1068.73	3.8722
	Timoshenko (1921)	FSDT	135.81	0.8715	1066.70	3.8738
	Bernoulli-Euler	CBT	135.81	0.8646	1066.70	3.8738

Table 4. Comparison of displacements and stresses for three layered (0^0/*Core*/0^0) sandwich beam subjected to sinusoidal load.

L/h	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
4	Present	RBT	0.0230	10.808	4.5109	0.1414
	Reddy (1984)	HSDT	0.0230	10.815	4.5092	0.1453
	Timoshenko (1921)	FSDT	0.0137	6.7293	2.6899	0.1580
	Bernoulli-Euler	CBT	0.0137	1.3627	2.6899	0.1580
10	Present	RBT	0.2394	2.9852	18.803	0.2765
	Reddy (1984)	HSDT	0.2389	2.9834	18.761	0.3774
	Timoshenko (1921)	FSDT	0.2140	2.2214	16.812	0.3950
	Bernoulli-Euler	CBT	0.2140	1.3627	16.812	0.3950
20	Present	RBT	1.7674	1.7755	69.407	0.5293
	Reddy (1984)	HSDT	1.7626	1.7721	69.218	0.7812
	Timoshenko (1921)	FSDT	1.7124	1.5774	67.248	0.7901
	Bernoulli-Euler	CBT	1.7124	1.3627	67.248	0.7901
50	Present	RBT	26.960	1.4323	423.49	1.3063
	Reddy (1984)	HSDT	26.883	1.4284	422.28	1.9717
	Timoshenko (1921)	FSDT	26.757	1.3971	420.30	1.9753
	Bernoulli-Euler	CBT	26.757	1.3627	420.30	1.9753
100	Present	RBT	214.93	1.3831	1688.09	2.6078
	Reddy (1984)	HSDT	214.31	1.3792	1683.19	3.9489
	Timoshenko (1921)	FSDT	214.05	1.3713	1681.21	3.9507
	Bernoulli-Euler	CBT	214.05	1.3627	1681.21	3.9507

Table 5. Comparison of displacements and stresses for five layered ($0^0/90^0/Core/90^0/0^0$) sandwich beam subjected to sinusoidal load.

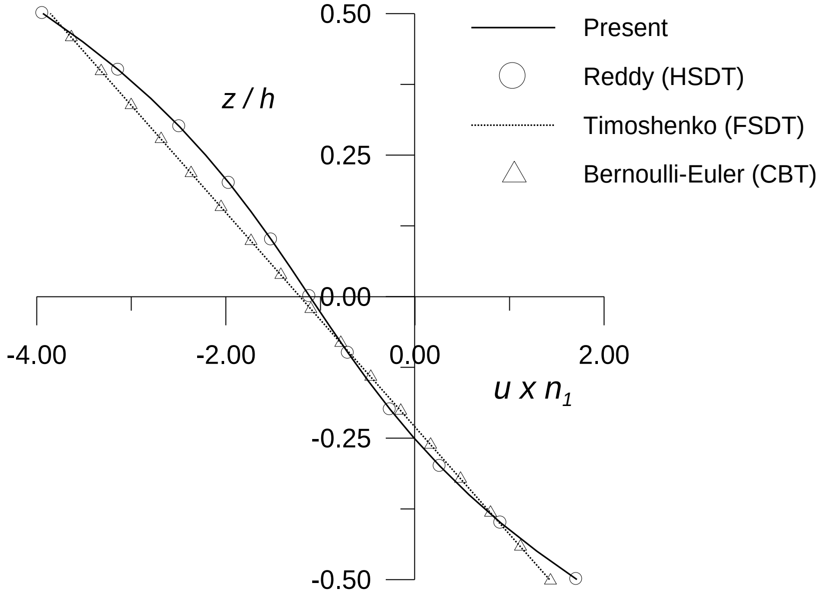

Fig. 2. Through thickness distribution of axial displacement ($\bar{u}$) for two layered ($0^0/90^0$) laminated composite beam subjected to sinusoidal load at L/h = 4.

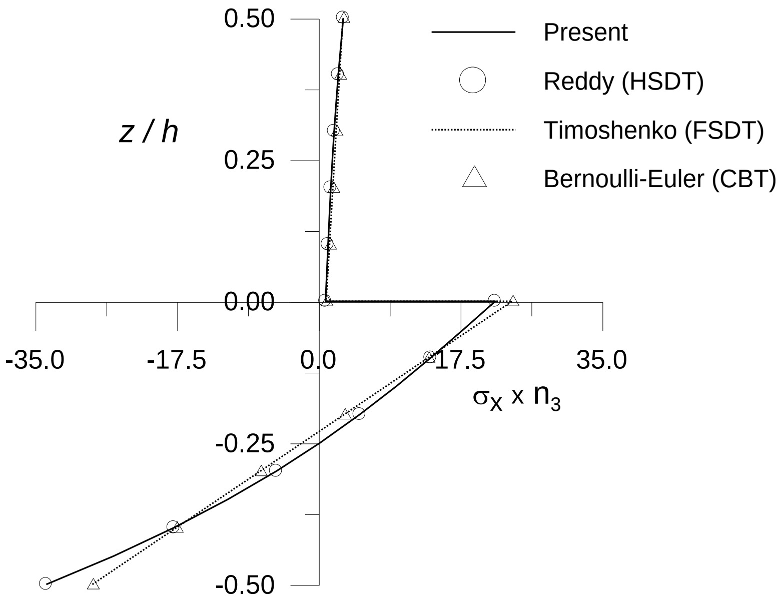

Fig. 3. Through thickness distribution of bending stress ($\bar{\sigma}_x$) for two layered ($0^0/90^0$) laminated composite beam subjected to sinusoidal load at L/h = 4.

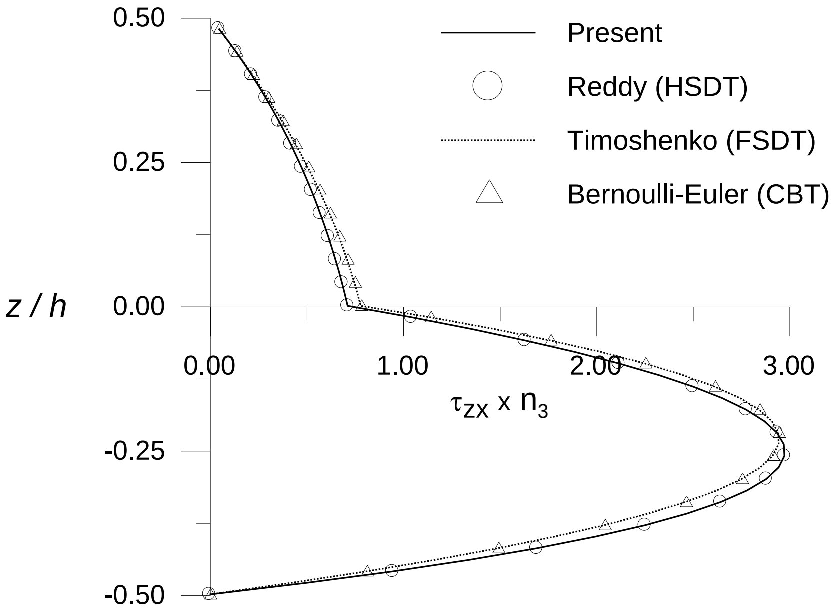

Fig. 4. Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for two layered ($0^0/90^0$) laminated composite beam subjected to sinusoidal load at $L/h = 4$.

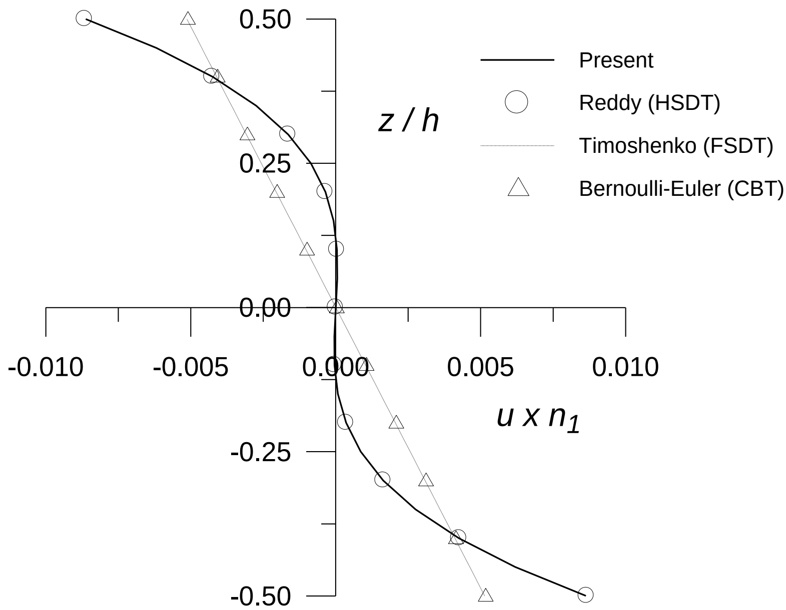

Fig. 5. Through thickness distribution of axial displacement ($\bar{u}$) for three layered ($0^0/90^0/0^0$) laminated composite beam subjected to sinusoidal load at $L/h = 4$.

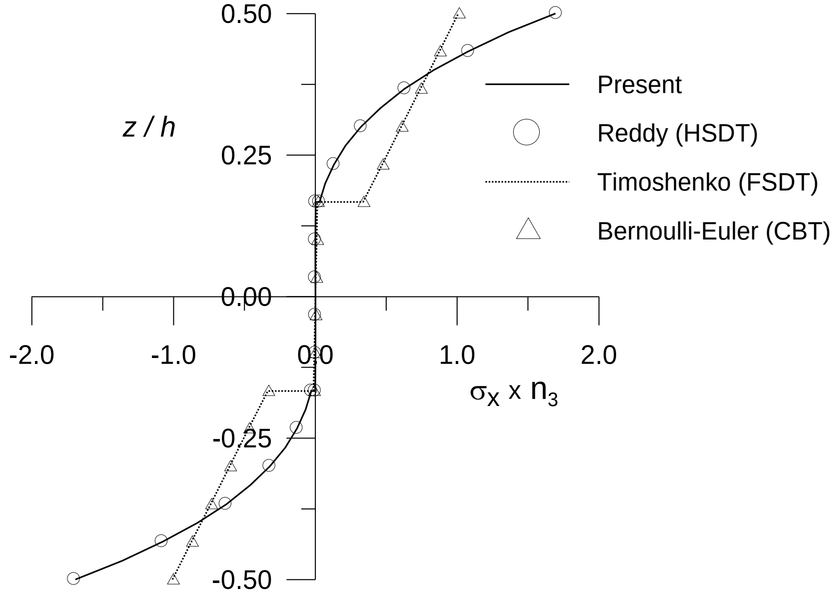

Fig. 6. Through thickness distribution of bending stress ($\bar{\sigma}_x$) for three layered ($0^0/90^0/0^0$) laminated composite beam subjected to sinusoidal load at L/h = 4.

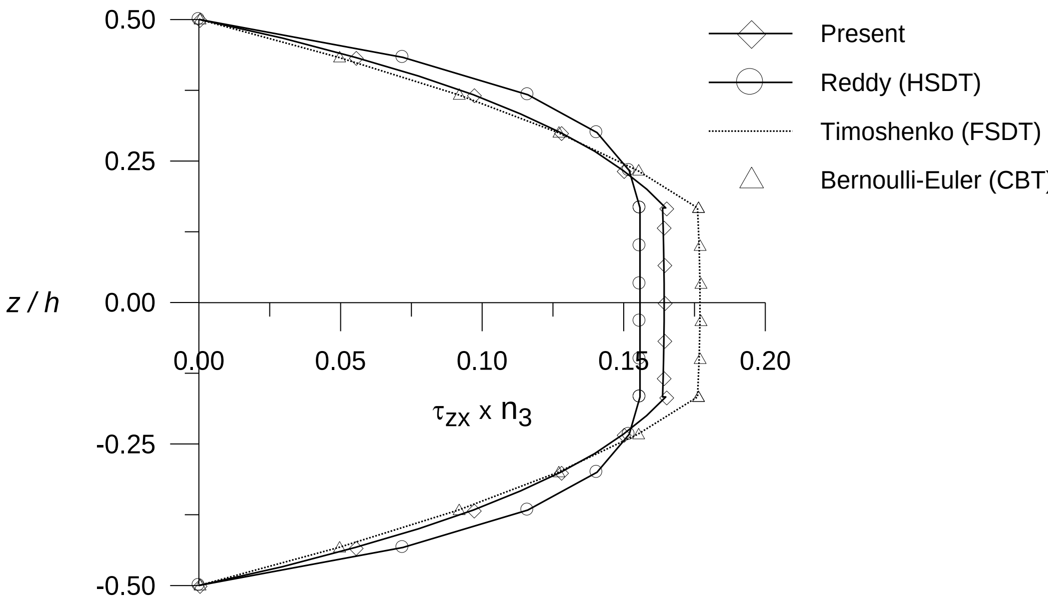

Fig. 7. Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for three layered ($0^0/90^0/0^0$) laminated composite beam subjected to sinusoidal load at L/h = 4.

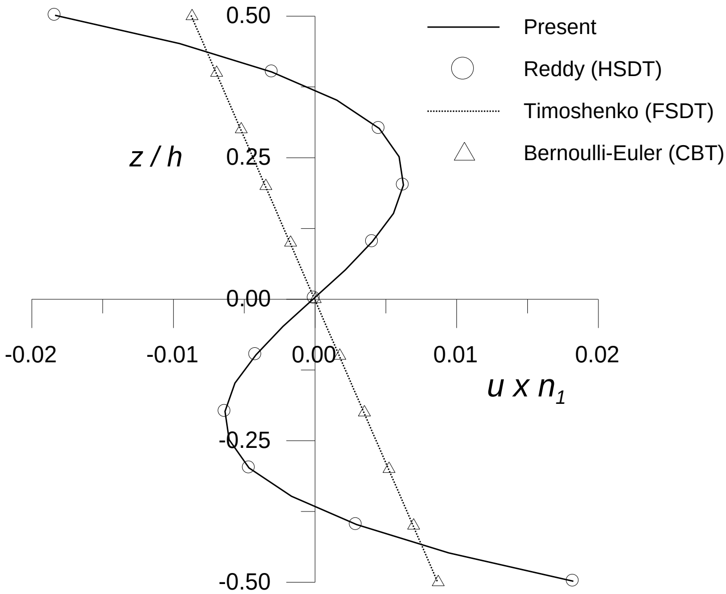

Fig. 8. Through thickness distribution of axial displacement ($\bar{u}$) for three layered (0^0/*core*/0^0) sandwich beam subjected to sinusoidal load at $L/h = 4$.

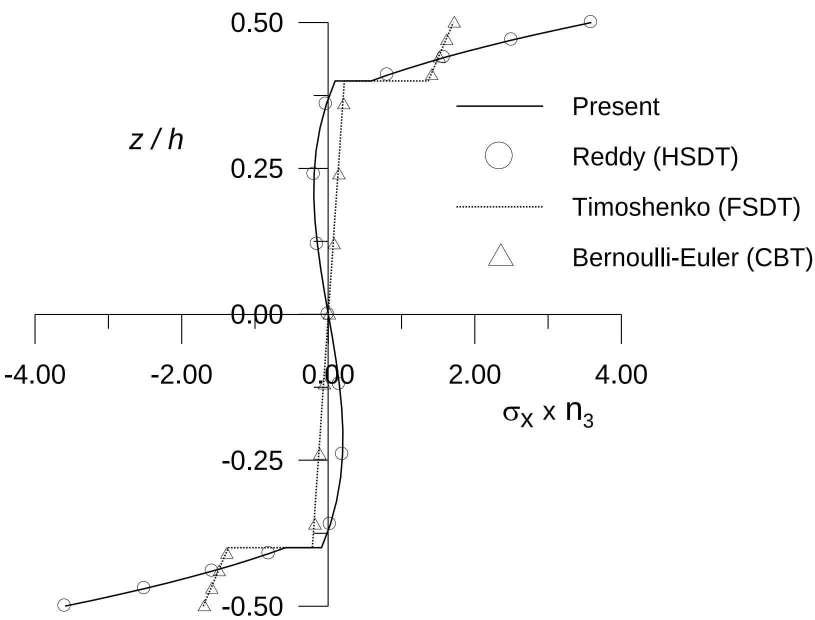

Fig. 9. Through thickness distribution of bending stress ($\bar{\sigma}_x$) for three layered (0^0/*core*/0^0) sandwich beam subjected to sinusoidal load at $L/h = 4$.

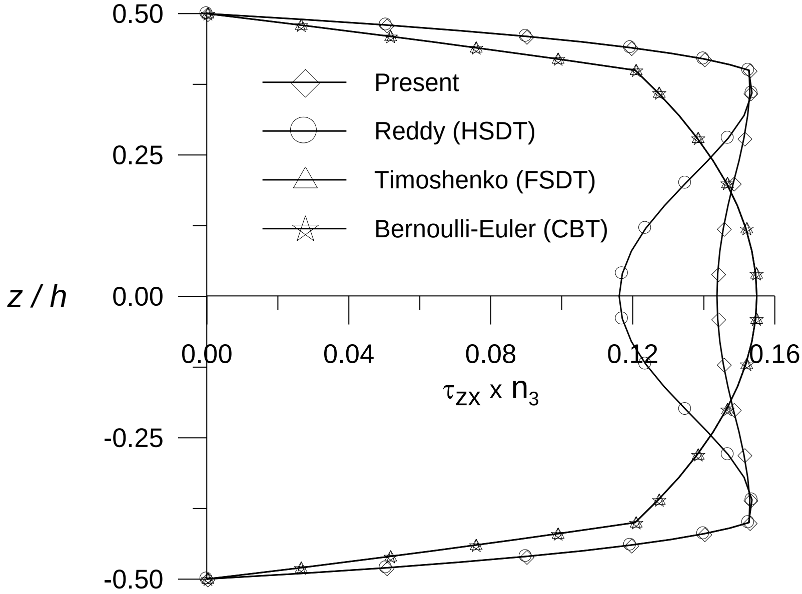

Fig. 10. Through thickness distribution of transverse shear stress ($\overline{\tau}_{zx}$) for three layered (0^0/*core*/0^0) sandwich beam subjected to sinusoidal load at L/h = 4.

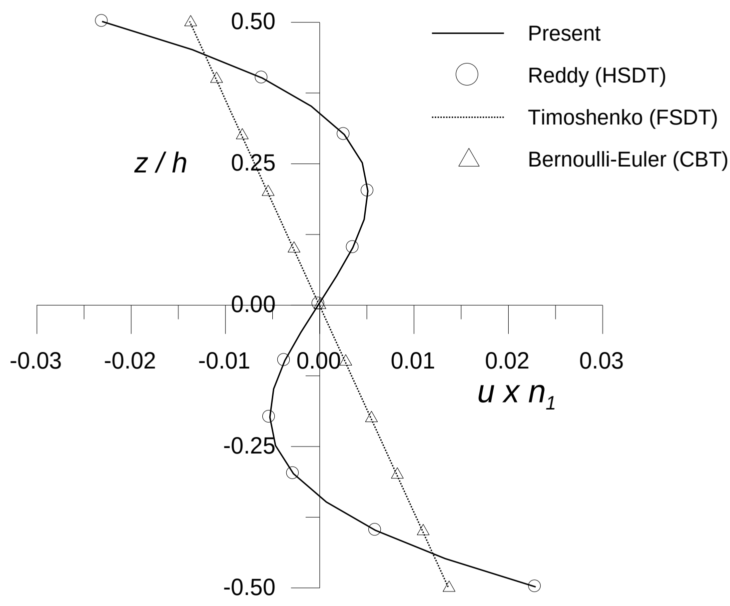

Fig. 11. Through thickness distribution of axial displacement ($\overline{u}$) for five layered ($0^0/90^0$/*core*/$90^0/0^0$) sandwich beam subjected to sinusoidal load at L/h = 4.

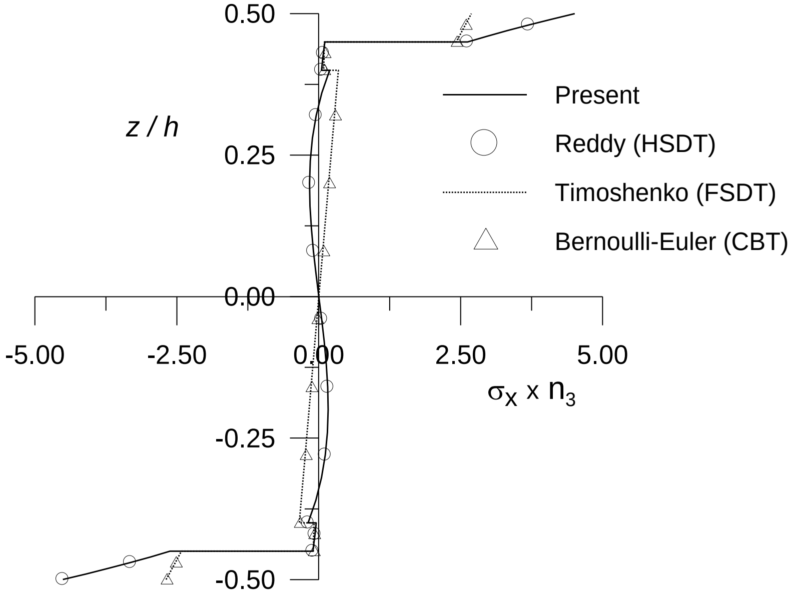

Fig. 12. Through thickness distribution of bending stress ($\bar{\sigma}_x$) for five layered ($0^0/90^0/core/90^0/0^0$) sandwich beam subjected to sinusoidal load at L/h = 4.

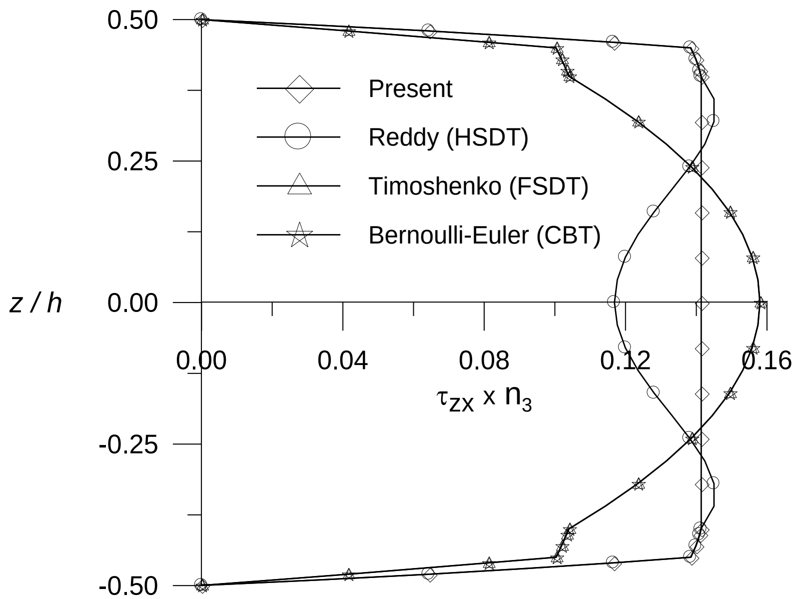

Fig. 13. Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for five layered ($0^0/90^0/core/90^0/0^0$) sandwich beam subjected to sinusoidal load at L/h = 4.

5. Conclusions

In this paper, the refined beam theory has been applied for laminated composite and soft core sandwich beams. The mathematical formulation and application of the present theory to bending analysis of beams led to the following conclusions:

1. The theory satisfies the zero transverse shear conditions on top and bottom surfaces of the beam. The transverse stress continuity is satisfied using equilibrium equations of the theory of elasticity.
2. The governing equations and boundary conditions are variationally consistent.
3. The theory obviates the need of shear correction factors which are generally associated with the first order shear deformation theory.
4. The present results are in excellent agreement with those of the exact solution and the HSDT of Reddy.
5. The CBT and the FSDT show inaccurate results compared with the present theory and the HSDT of Reddy.

Извод

Анализа напона код ламинарних композитних и сендвич греда са меким језгром уз помоћ теорије смичућег напона вишег реда

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Резиме

У раду се испитује побољшана теорија греда (РБТ) у светлу савијања просто ослоњених изотропних, ламинарних композита и сендвич греда. Осно поље померања користи параболичну функцију за ординату дебљине како би се укључио ефекат трансверзалне смичуће деформације. Трансверзално померање састоји се од савијајућих и смичућих компоненти. Садашња теорија задовољава тангенциону компоненту напона горњих и доњих површина греде без узимања у обзир проблемског смичућег корективног фактора Тимошенка. Главне диференцијалне једначине и гранични услови везани за претпостављено поље померања добијене су по принципу виртуелног рада. Како би доказали веродостојност теорије, применили смо је на анализу савијања греда. Просто ослоњени изотропни, ламинарни композити и сендвич греде анализирани су путем Навије приступа. Нумерички резултати недимензионалних померања и напона добијени уз помоћ садашње теорије представљени су и упоређени са резултатима побољшаних теорија доступних у литератури заједно са решењем еластичности.

Кључне речи: трансверзална смичућа деформација, смичући корективни фактор, трансверзални смичући напон, савијање, ламинарни композити, сендвич.

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