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Geometrically nonlinear analysis of laminated composite plates using a layerwise displacement model

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Abstract

In this paper the geometrically nonlinear laminated finite element model is developed using the principle of virtual displacements (PVD). The 3D elasticity equations are reduced to 2D problem using kinematical assumptions based on assumed layerwise displacement field of Reddy. With the assumed displacement field, nonlinear Green-Lagrange small strain large displacements relations and linear orthotropic material properties for each lamina, the PVD is used to obtain the weak form of the problem. The weak form or nonlinear integral equilibrium equations are discretized using isoparametric finite element approximation. The nonlinear incremental algebric equilibrium equations are solved using the direct iteration procedure. The original MATLAB computer program is coded for finite element solution and is used to investigate the geometrical nonlinear effects on displacement and stress field of thin and thick, isotropic, orthotropic and anisotropic laminated composite plates with various boundary conditions and the sign of the loading (loading/unloading). The accuracy of the numerical model is verified by comparison with results from the literature and the linear solutions from the previous paper. Appropriate conclusions are derived.

Key words: geometrically nonlinear analysis, layerwise finite element model

1. Introduction

During the last decade, there has been increasing use of composites in the design of primary load carrying members in aerospace and automotive industry, ship building industry and bridge design. The low mass density associated with high tensile strength provides them with high strength to weight ratios and high specific modulus. As a result of their lightness, composites replaced most traditional materials without being constrained in slenderness and thickness. The second outstanding feature of composite laminates is their so called “controlled anisotropy” associated with manufacturing flexibility one has to control mechanical properties of composite laminates by adjusting at will the lamina orientation in the stacking sequence of the laminate.

The above mentioned features resulted in large weight savings and made possible the use of very thin composite plate elements. However these elements become susceptible to large deflections during their service life (Polat et al. 2007, Zhang et al. 2006). In such cases the geometry of structure is continually changing during the deformation and geometrically nonlinear analysis should be adopted. The geometrically nonlinear analysis seems also to be necessary for obtaining the structural response of unsymmetrical laminated composite materials (Zhang et al. 2003). Namely, the nonlinear response of these laminates is present even for small displacements, due to complex coupling between in-plane and out-of plane deformation.

A considerable amount of research work has been carried out so far on the nonlinear analysis of laminated plates. Among the published works, the von Karman plate theory of plates undergoing large deflections has attracted outstanding attention and a number of papers have been published. The first authors investigating the nonlinear response using the von Karman nonlinear theory (Tanriover et al. 2004, Reddy et al. 1983) were: Leissa, Bennett, Bert, Chandra and Raju, Zaghloul and Kennedy, Chia and Prabhakara, Noor and Hartley, and in the last decades Han, Tabiei and Park, Singh, Lal and Kumar, Reddy and Chao, Zhang Kim and others.

Mechanical response of laminated composite material is generally 3D problem of nonlinear mechanics. However, due to its mathematical complexity, analytical solutions using 3D theory of elasticity are usually difficult and some times even impossible to achieve, while numerical solutions are computationally inefficient and constrained to very specific domains. Thus, whenever possible, refined simplified mathematical models, with acceptable accuracy in a field of applications, should be used. It is shown that the Equivalent Single Layer theories (ESL) may give acceptable results when analyzing global response, such as gross deflections and gross stresses, critical buckling loads and fundamental frequencies of thin to moderate thick laminated composite plates (Vuksanovic 2000). However, a continuous displacement function in ESL is not able to accurately present the discontinuous zigzag variation of displacements in highly anisotropic plates and give adequate stress distribution at local or ply level (Cetkovic et al. 2009). A compromise between 3D theory of elasticity and ESL theories is then achieved with the use of Layer Wise theories (LW). In LW theories the in-plane displacement field, assumed for each layer, is interpolated through the thickness by appropriate layerwise Lagrange interpolation function or Heaviside step function (Reddy 2004), thus replacing 3D laminated element with N+1 2D plate elements (N is number of layers), which fulfills the continuity of displacement functions at the interfaces between adjacent layers.

From the continuum mechanics it is known that two different level of geometrical nonlinearity may be modeled, which are: geometrically nonlinear models with small strain and large displacements (von Karman theory) and geometrically nonlinear models with large strains. In the first case, the geometry of the structure before deformation remains unchanged after the deformation. However, the structure is subjected to large displacements and the equilibrium is achieved on the configuration displaced from the undeformed one. In the second case the geometry of the structure is changing during the deformation and the equilibrium is achieved on the deformed configuration. In both cases equilibrium equations are nonlinear.

In order to formulate nonlinear finite element model of laminated structures, which will be able to represent two above mentioned levels of geometrical nonlinearity, two distinct approaches have been reported in the literature (Reddy 2004). The first approach is based on laminate theory, in which 3D elasticity equations are reduced to 2D equations through certain kinematical assumptions and homogenization through the thickness. In this approach only first type of nonlinearity or small strain, large displacement assumption may be included. The finite elements based on such an assumptions are named the laminated elements. The second approach is based on 3D continuum formulation (total and updated Lagrange formulation) and both types on nonlinearity may be included. Finite elements based on this approach are called the continuum elements.

The aim of the author's research on composite materials so far was to implement Layerwise theory of Reddy or Generalized Layerwise Plate Theory-GLPT (Reddy et al. 1989) on different levels of analysis of laminated composite plates. The previous work has been concerned with the linear analysis (Cetkovic et al. 2009), and the linear laminated plate element of GLPT has been formulated, while in the present paper the GLPT nonlinear laminated plate element with von Karman geometrical nonlinearity is presented.

In this paper the mathematical and numerical model for geometrically nonlinear, small strain, large displacements problem of laminated composite plates is presented. The 3D elasticity equations are reduced to 2D problem using kinematical assumptions based on layerwise displacement field of Reddy (GLPT). With the assumed displacement field, nonlinear Green-Lagrange small strain large displacements relations and linear orthotropic material properties for each lamina, the principle of virtual displacement (PVD) is used to derive the weak form of the problem. The weak form or nonlinear integral equilibrium equations are discretized using isoparametric finite element approximation. The obtained nonlinear incremental algebric equilibrium equations are solved using direct iteration procedure. The originally coded MATLAB computer program for the finite element solution is used to investigate the effects of geometrical nonlinearity on displacement and stress field of thin and thick, isotropic, orthotropic and anisotropic laminated composite plates with various boundary conditions and the sign of the loading (loading/unloading). The accuracy of the numerical model is verified by being compared with available results from the literature and the linear solutions from the previous paper (Cetkovic et al. 2009). The appropriate conclusions are derived.

2. Theoretical formulation

2.1 Displacement field

In the LW theory of Reddy (Reddy et al. 1989) or Generalized Layerwise Plate Theory (GLPT), in-plane displacements components (u, v) are interpolated through the thickness using 1D linear Lagrangian interpolation function $\Phi^I(z)$, while transverse displacement component w is assumed to be constant through the plate thickness.

$$\begin{aligned} u_1(x,y,z) &= u(x,y) + \sum_{I=1}^{N+1} U^I(x,y)\cdot\Phi^I(z) \\ u_2(x,y,z) &= v(x,y) + \sum_{I=1}^{N+1} V^I(x,y)\cdot\Phi^I(z) \\ u_3(x,y,z) &= w(x,y) \end{aligned} \tag{1}$$

thus giving the "zig-zag" or layer wise variation of the in-plane displacements. This "zig-zag" behavior is more pronounced for thick laminates, where the transverse shear modulus change abruptly through the thickness and can be seen in the exact 3D elasticity solutions obtained by Pagano, Srinavas and Rao, Noor etc. Therefore, layerwise displacement fields provide a much more kinematically correct representation of the moderate to severe cross sectional warping associated with the deformation of thick laminates.

2.2 Strain-displacement relations

The Green Lagrange strain tensor associated with the displacement field Eq.(1) can be computed using von Karman strain-displacement relation to include geometric nonlinearities as follows:

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \sum_{I=1}^{N+1}\frac{\partial U^I}{\partial x}\Phi^I + \underline{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2}$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \sum_{I=1}^{N+1}\frac{\partial V^I}{\partial y}\Phi^I + \underline{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N+1}\left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x}\right)\Phi^I + \underline{\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}} \tag{2}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \sum_{I=1}^{N+1} U^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \sum_{I=1}^{N+1} V^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial y}$$

2.3 Constitutive equations

For Hook's elastic material, the stress-strain relations for k-th orthotropic lamina have the following form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^{(k)} \times \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)} \tag{3}$$

where $\boldsymbol{\sigma}^{(k)} = \{\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}\}^{(k)T}$ and $\boldsymbol{\varepsilon}^{(k)} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz}\}^{(k)T}$ are stress and strain components respectively, and $\mathbf{Q}_{ij}^{(k)}$ are transformed elastic coefficients, of k-th lamina in global coordinates.

2.4 Equilibrium equations

Equilibrium equations may be obtained from the Principle of Virtual Displacements (PVD), in which sum of external virtual work done on the body and internal virtual work stored in the body should be equal zero:

$$0 = \int_{\Omega}\left[\left(\{\delta\varepsilon^0\}^T + \{\delta\varepsilon^m\}^T\right)\{N^0\} + \{\delta\varepsilon^I\}^T\{N^I\} + \delta u\, q_x^0 + \delta v\, q_y^0 + \delta w\, q_z^0\right]dxdy$$
$$-\oint_{\Gamma}\delta u_n N_{nn}\, ds - \oint_{\Gamma}\delta u_s N_{ns}\, ds - \oint_{\Gamma}\delta w\left(Q_n + P_n\right)ds - \oint_{\Gamma}\delta U_n^I N_{nn}^I\, ds - \oint_{\Gamma}\delta U_s^I N_{ns}^I ds \tag{4}$$

where $\left\{q_x^0, q_y^0, q_z^0\right\}$ is distributed load in x, y, z directions, while internal forces are:

$$\begin{Bmatrix} \{\mathbf{N}^0\} \\ \{\mathbf{N}^I\} \end{Bmatrix} = \begin{bmatrix} [\mathbf{A}] & [\mathbf{B}^I] \\ [\mathbf{B}^I] & \sum_{J=1}^{N} [\mathbf{D}^{JI}] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{\varepsilon}^0\} + \{\boldsymbol{\varepsilon}^m\} \\ \{\boldsymbol{\varepsilon}^I\} \end{Bmatrix} \tag{5}$$

where **A**, **B**, $\mathbf{B^I}$, $\mathbf{D^{JI}}$ matrices are given in (Cetkovic 2005), while internal force vectors are:

$$\{\mathbf{N}^0\} = \{N_{xx} \quad N_{yy} \quad N_{xy} \quad Q_x \quad Q_y\}^T, \qquad \{\mathbf{N}^I\} = \{N_{xx}^I \quad N_{yy}^I \quad N_{xy}^I \quad Q_x^I \quad Q_y^I\}^T \tag{6$_{1,2}$}$$

$$N_{nn} = N_{xx} n_x + N_{xy} n_y, \qquad N_{ns} = N_{xy} n_x + N_{yy} n_y, \qquad Q_n = Q_x n_x + Q_y n_y \tag{7$_{1,2,3}$}$$

$$P_n = \left(N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) n_x + \left(N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) n_y \tag{8}$$

$$N_{nn}^I = N_{xx}^I n_x + N_{xy}^I n_y, \qquad N_{ns}^I = N_{xy}^I n_x + N_{yy}^I n_y \tag{9}$$

and strain vectors are:

$$\{\boldsymbol{\varepsilon}^0\} = \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \right\}^T$$

$$\{\boldsymbol{\varepsilon}^m\} = \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \quad 0 \quad 0 \right\}^T$$

$$\{\boldsymbol{\varepsilon}^I\} = \left\{ \frac{\partial U^I}{\partial x} \quad \frac{\partial V^I}{\partial y} \quad \frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x} \quad U^I \quad V^I \right\}^T \tag{10$_{1,2,3}$}$$

3. Finite Element Model

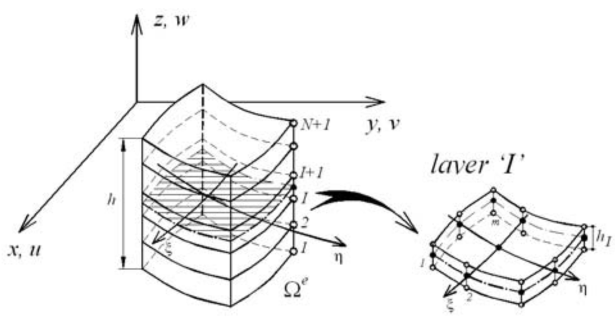

Fig. 1. Plate finite element with n layers and m nodes.

The GLPT finite element consists of middle surface plane and I=1, N+1 planes through the plate thickness Fig. 1. The element requires only the C^0 continuity of major unknowns, thus in each node only displacement components are adopted, that are (u, v, w) in the middle surface element nodes and (U^I, V^I) in the I-th plane element nodes. The generalized displacements over element Ω^e can be expressed as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} u_j \Psi_j \\ \sum_{j=1}^{m} v_j \Psi_j \\ \sum_{j=1}^{m} w_j \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\boldsymbol{\Psi}_j]^e \{\mathbf{d}_j\}^e \qquad \begin{Bmatrix} U^I \\ V^I \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} U_j^I \Psi_j \\ \sum_{j=1}^{m} V_j^I \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\overline{\boldsymbol{\Psi}}_j]^e \{\mathbf{d}_j^I\}^e \tag{11}$$

where $\{\mathbf{d}_j\}^e = \{u_j^e \quad v_j^e \quad w_j^e\}^T, \{\mathbf{d}_j^I\}^e = \{U_j^I \quad V_j^I\}^T$ are displacement vectors, in the middle plane and I-th plane, respectively, Ψ_j^e are interpolation functions, while $[\boldsymbol{\Psi}_j]^e, [\overline{\boldsymbol{\Psi}}_j]^e$ are interpolation function matrix for the j-th node of the element Ω^e, given in (Cetkovic et al. 2009). Substituting element displacement field Eq.(6) in to weak form Eq.(4), the nonlinear laminated finite element is obtained (Cetkovic et al. 2011):

$$[\mathbf{K}_{NL}]^e \cdot \{\mathbf{d}\}^e = \{\mathbf{f}\}^e \tag{12}$$

where secant stiffness matrix is:

$$[\mathbf{K}_{NL}]^e = \begin{bmatrix} [\mathbf{K}^{11}]^e & [\mathbf{K}^{12}]^e \\ [\mathbf{K}^{12}]^e & [\mathbf{K}^{22}]^e \end{bmatrix}$$

$$[\mathbf{K}^{11}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} [\mathbf{H}_i^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_j^e] + \underline{[\mathbf{H}_i^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_{jNL}^e]} + \underline{2[\mathbf{H}_{iNL}^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_j^e]} + \underline{2[\mathbf{H}_{iNL}^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_{jNL}^e]}\, d\Omega^e$$

$$[\mathbf{K}^{12}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} \left([\mathbf{H}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\overline{\mathbf{H}}_j^e] + \underline{2\,[\mathbf{H}_{iNL}^e]^T \cdot [\mathbf{B}^I] \cdot [\overline{\mathbf{H}}_j^e]} \right) d\Omega^e$$

$$[\mathbf{K}^{21}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} \left([\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\mathbf{H}_j^e] + \underline{[\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\mathbf{H}_{jNL}^e]} \right) d\Omega^e \tag{13$_{1,2,3,4}$}$$

$$[\mathbf{K}^{22}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} [\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{D}^{IJ}] \cdot [\overline{\mathbf{H}}_j^e]\, d\Omega^e$$

and external force vector $\{\mathbf{f}\}^e = \begin{Bmatrix} \{\mathbf{f}^0\}^e \\ \{\mathbf{f}^I\}^e \end{Bmatrix}$ is:

$$\{\mathbf{f}^0\}^e=\sum_{i=1}^{m}\left[\int_{\Omega^e}[\mathbf{\Psi}_i^e]^T\begin{Bmatrix}q_x^0\\ q_y^0\\ q_z^0\end{Bmatrix}d\Omega^e+\oint_{\Gamma^e}[\mathbf{\Psi}_i^e]^T\begin{Bmatrix}N_{nn}\\ N_{ns}\\ Q_n+P_n\end{Bmatrix}d\Gamma^e\right]$$

$$\{\mathbf{f}^I\}^e=\sum_{i=1}^{m}\left[\int_{\Omega^e}[\overline{\mathbf{\Psi}}_i^e]^T\cdot\begin{Bmatrix}q_x^I\\ q_y^I\end{Bmatrix}d\Omega^e+\oint_{\Gamma^e}[\overline{\mathbf{\Psi}}_i^e]^T\cdot\begin{Bmatrix}N_{nn}^I\\ N_{ns}^I\end{Bmatrix}d\Gamma^e\right] \qquad (14)_{1,2}$$

while:

$$[\mathbf{H}_j^e]_j^e=\begin{bmatrix}\frac{\partial\Psi_j^e}{\partial x} & 0 & 0\\ 0 & \frac{\partial\Psi_j^e}{\partial y} & 0\\ \frac{\partial\Psi_j^e}{\partial y} & \frac{\partial\Psi_j^e}{\partial x} & 0\\ 0 & 0 & \frac{\partial\Psi_j^e}{\partial x}\\ 0 & 0 & \frac{\partial\Psi_j^e}{\partial y}\end{bmatrix},\quad [\mathbf{H}_{j\,NL}^e]=\frac{1}{2}\begin{bmatrix}0 & 0 & \frac{\partial w}{\partial x}\frac{\partial\Psi_j^e}{\partial x}\\ 0 & 0 & \frac{\partial w}{\partial y}\frac{\partial\Psi_j^e}{\partial y}\\ 0 & 0 & \frac{\partial w}{\partial x}\frac{\partial\Psi_j^e}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial\Psi_j^e}{\partial x}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$[\overline{\mathbf{H}}_j^e]_j^e=\begin{bmatrix}\frac{\partial\Psi_j^e}{\partial x} & 0\\ 0 & \frac{\partial\Psi_j^e}{\partial y}\\ \frac{\partial\Psi_j^e}{\partial y} & \frac{\partial\Psi_j^e}{\partial x}\\ 1 & 0\\ 0 & 1\end{bmatrix} \qquad (15)_{1,2,3}$$

With the known displacement field, the stress field over the element may be obtained as a part of a postprocessor, using strain displacement and constitutive relations, Eqs. (2), (3) as:

$$\{\boldsymbol{\sigma}_b\}_U^{(\kappa)\,e}=[\mathbf{Q_b}]^{(k)}\sum_{j=1}^{m}\left([\mathbf{H_{bj}}]+[\mathbf{H_{bj}^{NL}}]\right)\{\mathbf{d}_j\}^e+[\mathbf{Q_b}]^{(k)}\sum_{j=1}^{m}[\overline{\mathbf{H}}_{\mathbf{b}_j}]\{\mathbf{d}_j^I\}^e$$

$$\{\boldsymbol{\sigma}_b\}_O^{(\kappa)\,e}=[\mathbf{Q_b}]^{(k)}\sum_{j=1}^{m}\left([\mathbf{H_{bj}}]+[\mathbf{H_{bj}^{NL}}]\right)\{\mathbf{d}_j\}^e+[\mathbf{Q_b}]^{(k)}\sum_{j=1}^{m}[\overline{\mathbf{H}}_{\mathbf{b}_j}]\{\mathbf{d}_j^{I+1}\}^e$$

$$\{\boldsymbol{\sigma}_s\}_{const}^{(k)\,e}=[\mathbf{Q_s}]^{(k)}\sum_{j=1}^{m}[\mathbf{H_{sj}}]\{\mathbf{d}_j\}^e+[\mathbf{Q_s}]^{(k)}\sum_{j=1}^{m}[\overline{\mathbf{H}}_{\mathbf{sj}}]\left(\{\mathbf{d}_j^{I+1}\}^e-\{\mathbf{d}_j^I\}^e\right)/h_k \qquad (16)_{1,2,,3}$$

where $\{\boldsymbol{\sigma}_b\}_U^{(\kappa)\,e}$ and $\{\boldsymbol{\sigma}_b\}_O^{(\kappa)\,e}$ are in-plane normal stresses $(\sigma_{xx}, \sigma_{yy}, \tau_{xy})$ at bottom and upper plane in k-th layer of plate element 'e', while $\{\boldsymbol{\sigma}_s\}_{const}^{(k)\ \ e}$ are average transverse shear stresses (τ_{xz}, τ_{yz}) in k-the layer of plate element.

4. Numerical results and discussion

Based on the previously derived laminated finite element model for the geometrically nonlinear analysis of laminated composite plates, the original computer program is coded using MATLAB programming language. The quadratic Lagrange rectangular element with nine nodes and associated polynomial was used for isoparametric FE approximation of in-plane displacement of plate element and geometry. The nonlinear finite element secant stiffness matrix is evaluated using Gauss–Legendre quadrature rule, which are 3x3 Gauss integration schemes or 2D quadratic Lagrange rectangular element for in-plane interpolation and 1D linear Lagrange element for through the thickness interpolation. The direct iteration procedure, also known as the Picard iteration method was used as numerical procedure to solve nonlinear algebric equations, iterative in nature. The effects of plate thickness, lamination scheme, boundary conditions and the sign of the loading on nonlinear response of isotropic, orthotropic and anisotropic plates are analyzed. The accuracy of the present formulation is demonstrated through a number of examples and by comparison with results available from the literature.

The following boundary conditions at the plate edges are analyzed (Thankam et al. 2003). Simply supported (SS):

$$\textbf{SS:}\quad \begin{cases} x=0,a: & v_0 = w_0 = V^I = N_{xx} = N_{xx}^I = 0 \\ y=0,b: & u_0 = w_0 = U^I = N_{yy} = N_{yy}^I = 0 \end{cases} \quad I = 1,\ldots N+1 \tag{17}$$

Simply supported-hinged (HH):

$$\textbf{HH}:\quad \begin{cases} x=0,a: & u_0 = v_0 = w_0 = V^I = N_{xx}^I = 0 \\ y=0,b: & u_0 = v_0 = w_0 = U^I = N_{yy}^I = 0 \end{cases} \quad I = 1,\ldots N+1 \tag{18}$$

Clamped (CC):

$$\textbf{CC:}\quad \begin{cases} x=0,a: & u_0 = v_0 = w_0 = U^I = V^I = 0 \\ y=0,b: & u_0 = v_0 = w_0 = U^I = V^I = 0 \end{cases} \quad I = 1,\ldots N+1 \tag{19}$$

When analyzing a quarter of a plate, boundary conditions in the plane of symmetry become:

For cross ply laminates:

$$\textbf{SS1}:\quad \begin{cases} x=a/2: & u_0 = U^I = N_{yy} = N_{yy}^I = 0 \\ y=b/2: & v_0 = V^I = N_{xx} = N_{xx}^I = 0 \end{cases} \quad I = 1,\ldots N+1 \tag{20}$$

For angle ply laminates:

$$\textbf{SS2:}\quad \begin{cases} x=a/2: & v_0 = U^I = N_{xx} = N_{yy}^I = 0 \\ y=b/2: & u_0 = V^I = N_{yy} = N_{xx}^I = 0 \end{cases} \quad I = 1,\ldots N+1 \tag{21}$$

Example 4.1. A nonlinear bending of square, simply supported (SS1), isotropic plate, with $a = b = 25.4\text{cm}$ and $h = 2.54\text{cm}$ made of material:

$$E = 5{,}37791\,N/cm\text{^}2, \quad \nu = 0.3 \tag{22}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\overline{P} = q_0 \cdot a^4 / (Eh^4)$, the incremental load vector is chosen to be:

$$\{\Delta P\} = \{6.25, 6.25, 12.5, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0\} \cdot \overline{P} \tag{23}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}8$. The displacements and stresses are given in following nondimensional form:

$$\overline{w} = w_0 \cdot Eh^3 / (q_0 \cdot a^4), \quad \overline{\sigma}_{xx} = \sigma_{xx} \cdot (a/h)^2 \cdot 1/E \tag{24}$$

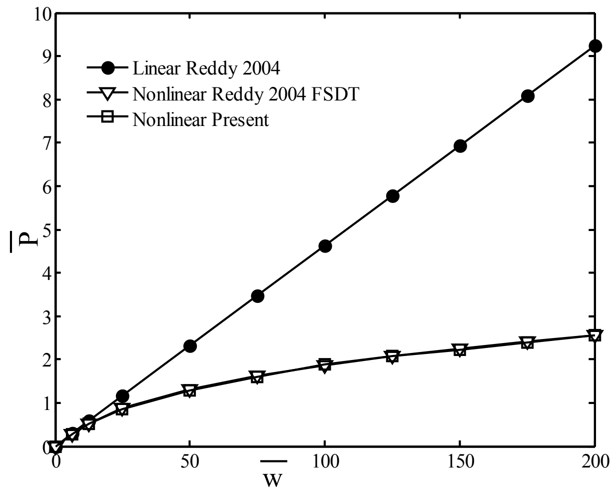

Fig. 2. Nonlinear bending of square simply supported (SS1) isotropic plate with $a/h = 10$; central displacement versus load parameter.

A 3x3 quarter plate laminated GLPT model is compared with 4x4 quadratic FSDT model [Reddy 2004]. The results for linear and nonlinear deflections are presented on Fig. 2. It is shown that proposed GLPT model closely agree with FSDT model. The Fig. 2 also demonstrates the physical nature of geometrically nonlinear response. The study has proved that depending of applied load level, the plate goes from the state of pure bending, at small displacement ($w \leq 0.30h$) to the phase of bending-stretching coupling, at large displacements. Namely, when the lateral displacement reaches approximately one half of plate thickness ($w \approx 0.5.h$), they take part in stretching, together with bending of the plate middle surface (nonlinear terms in Eq.(2)). This activates the tensile forces, thus enlarging the stiffness of the plates, and reducing displacements from the values predicted by linear theory. This may be the reason why this phenomena is also known as “plate stiffening” or “stress relaxation”. Moreover, the activation of tensile forces in laminated composite plates is of utmost importance, due to their high available specific tensile strength.

Example 4.2. A nonlinear bending of square simply supported (SS1), orthotropic plate made of high modulus glass-epoxy fiber reinforced material:

$$E_1/E_2 = 25, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, G_{23}/E_2 = 0.2,$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{25}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\overline{P} = q_0 \cdot a^4 / (E_2 h^4)$, the incremental load vector is chosen to be:

$$\{\Delta P\} = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140\} \cdot \overline{P} \tag{26}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}3$. The displacements and stresses are given in following nondimensional form:

$$\overline{w} = w_0 \cdot E_2 h^3 / (q_0 \cdot a^4)$$

$$(\overline{\sigma}_{xx}, \overline{\sigma}_{yy}, \overline{\tau}_{xy}) = (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) \cdot \left(\frac{h}{a}\right)^2 \cdot \frac{1}{E_2}, \quad \overline{\tau}_{xz} = \tau_{xz} \cdot \frac{h}{a} \cdot \frac{1}{E_2} \tag{27$_{1,2}$}$$

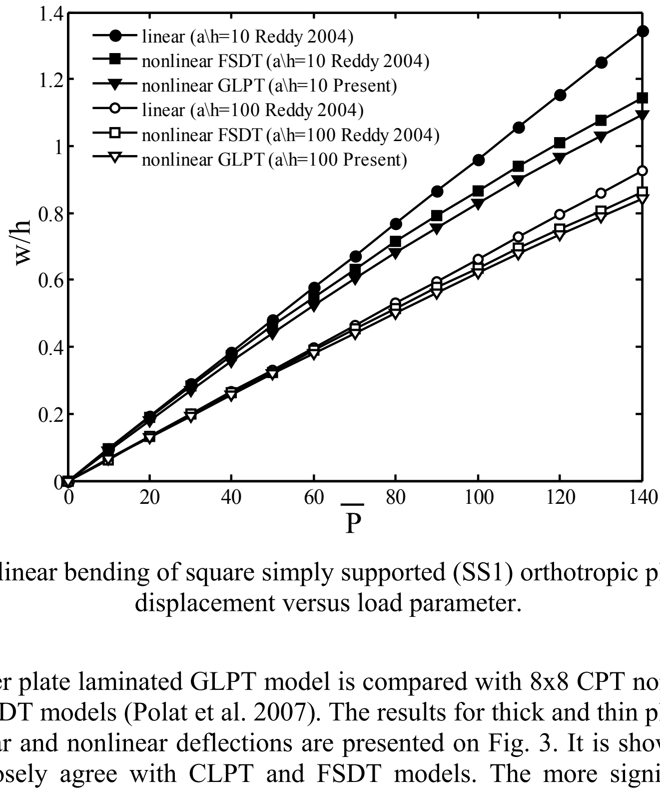

Fig. 3. Nonlinear bending of square simply supported (SS1) orthotropic plate; central displacement versus load parameter.

A 2x2 quarter plate laminated GLPT model is compared with 8x8 CPT nonconforming and 4x4 quadratic FSDT models (Polat et al. 2007). The results for thick and thin plates (a/h=10 and a/h=100) of linear and nonlinear deflections are presented on Fig. 3. It is shown that proposed GLPT model closely agree with CLPT and FSDT models. The more significant difference between linear and nonlinear solutions is observed for thick plates, while in thick plates larger lateral deflections have greater influence on nonlinear response, as it can be seen from the underlined nonlinear terms in Eq. (2).

Example 4.3. A nonlinear bending of square cross ply 0/90 and angle ply 45/-45 plates, with $a = b = 1$ and $h = 0.1$, with three different boundary conditions (SS, SS1 HH and CC, Eqs. 17, 18, 19, 20) , made of material:

$$E_1 / E_2 = 40, G_{12} / E_2 = 0.6, G_{13} / E_2 = 0.6, G_{23} / E_2 = 0.5, \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \quad (28)$$

subjected to uniform transverse pressure $\overline{q} = q(x,y) \cdot \left(\frac{a}{h}\right)^4 \cdot \frac{1}{E_2}$ are analyzed. The incremental load vector is:

$$\{\Delta\overline{q}\} = \{-100, -20, -20, -20, -20, 40, 20, 20, 20, 20\} \quad (29)$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}5$. The displacements and stresses are given in following nondimensional form:

$$\overline{w}_{LIN} = w \times \frac{h^3}{a^4} \frac{E_2}{q} \cdot 100, \quad (\overline{\sigma}_{xx}, \overline{\sigma}_{yy}) = (\sigma_{xx}, \sigma_{yy}) \times \left(\frac{a}{h}\right)^2 \cdot \frac{1}{E_2} \quad (30)$$

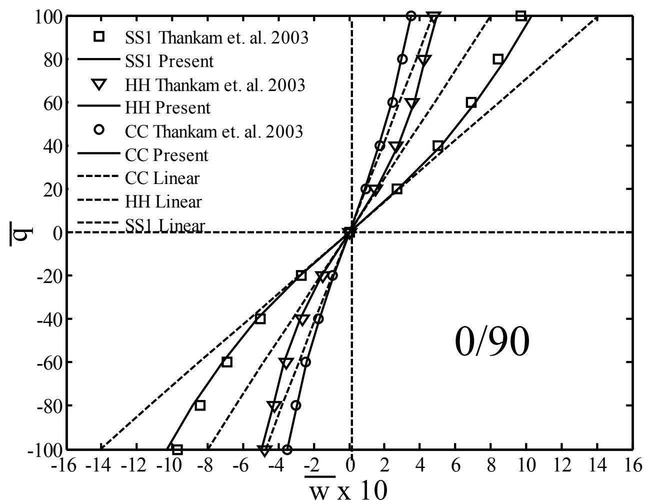

Fig. 4. Nonlinear bending of square cross ply 0/90 plate with different boundary conditions and $a / h = 10$; central displacement versus load parameter.

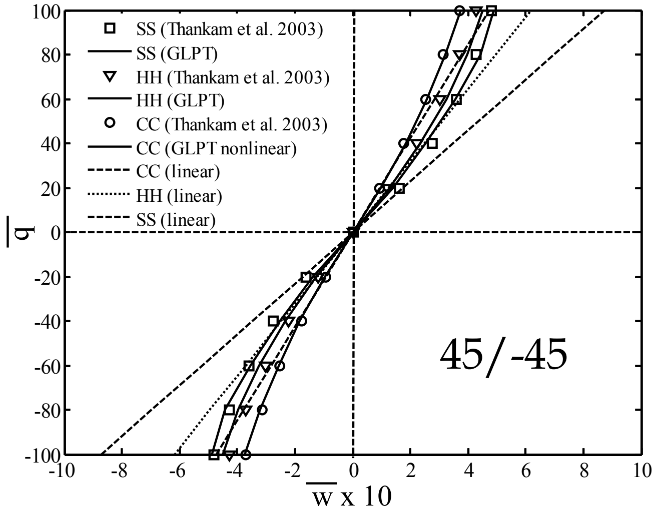

Fig. 5. Nonlinear bending of square angle ply 45/-45 plate with different boundary conditions and $a/h = 10$; central displacement versus load parameter.

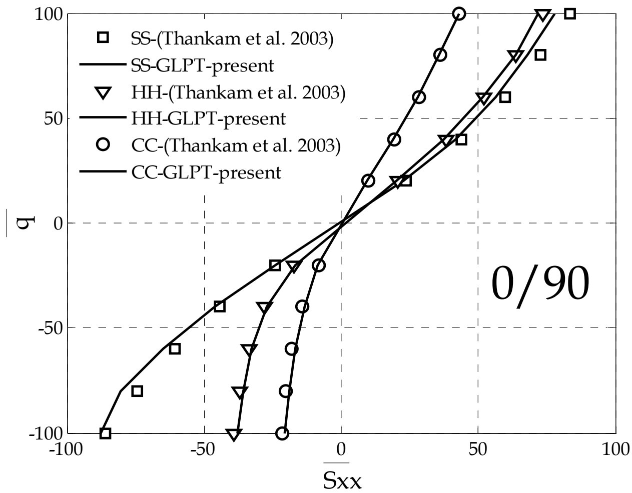

Fig. 6. Nonlinear bending of square cross ply 0/90 plate with different boundary conditions and $a/h = 10$; in plane stress $\overline{S}_{xx}$ versus load parameter.

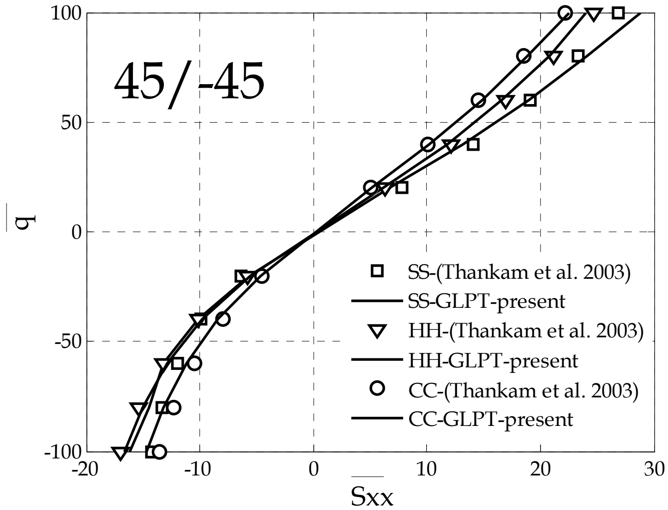

Fig. 7. Nonlinear bending of square angle ply 45/-45 plate with different boundary conditions and $a/h=10$; in plane stress $\overline{S}_{xx}$ versus load parameter.

$\overline{q}$	$\overline{\sigma}_{XX}$			$\overline{\sigma}_{YY}$		
	Thankam et al. 2003	Present full plate	Present quarter plate	Thankam et al. 2003	Present full plate	Present quarter plate
−100	−85.81	−87.7687	−91.4865	6.389	−6.2803	−6.1190
−80	−74.23	−80.2677	−83.2767	5.546	−5.4995	−5.3706
−60	−60.66	−64.9796	−66.6366	4.551	−4.5429	−4.4573
−40	−44.24	−46.1771	−47.1425	3.334	−3.3316	−3.3083
−20	−23.93	−24.2330	−24.4841	1.814	−1.8280	−1.8198
20	23.76	22.8399	22.5550	1.827	1.8946	1.8201
40	43.66	41.5389	40.5038	3.379	3.5618	3.3021
60	59.57	56.2325	54.1600	4.635	4.9345	4.4838
80	72.59	68.1087	65.2326	5.674	6.0725	5.4087
100	83.61	77.9324	74.6326	6.559	6.5274	6.1561
$\overline{w}_{LIN}$	1.2370	1.2115	1.2114	0.0940	0.0960	0.0960

Table 1. Stresses versus load parameter of square simply-supported (SS, SS1) orthotropic plate $a/h=10$.

$\bar{q}$	$\bar{\sigma}_{XX}$			$\bar{\sigma}_{YY}$		
	Thankam et al. 2003	Present		Thankam et al. 2003	Present	
		full plate	quarter plate		full plate	quarter plate
−100	−39.0700	−37.8815	−37.1852	−2.5720	−2.7585	−2.7143
−80	−37.0900	−35.6542	−35.2788	−2.3440	−2.4965	−2.4756
−60	−33.7000	−32.8219	−32.0224	−2.0350	−2.1942	−2.1512
−40	−27.7900	−26.8872	−26.3827	−1.5940	−1.7097	−1.6858
−20	−17.2700	−16.6923	−16.5961	−0.9330	−1.0001	−0.9954
20	20.4900	20.1643	20.2555	0.9570	1.0238	1.0249
40	38.0800	37.6015	38.0757	1.6730	1.7686	1.7877
60	52.1200	51.5368	52.4040	2.1860	2.2869	2.3170
80	63.6200	63.6309	64.2721	2.5760	2.7131	2.7150
100	73.3600	71.8442	74.4556	2.8940	2.9282	3.0357
$\bar{w}_{LIN}$	1.0100	0.9847	0.9848	0.0510	0.0545	0.0545

Table 2. Central displacement and stresses versus load parameter of square hinged (HH) cross ply 0/90 plate with $a / h = 10$.

$\bar{q}$	$\bar{\sigma}_{XX}$			$\bar{\sigma}_{YY}$		
	Thankam et al. 2003	Present		Thankam et al. 2003	Present	
		full plate	quarter plate		full plate	quarter plate
−100	−21.4000	−20.5632	−19.2411	−2.1120	−2.1710	−2.2723
−80	−20.2300	−19.1365	−18.3254	−1.8910	−1.9245	−2.0278
−60	−18.0200	−16.8767	−16.4203	−1.5970	−1.6135	−1.7035
−40	−14.2500	−13.1940	−12.9932	−1.1970	−1.1990	−1.2737
−20	−8.3160	−8.4576	−8.4757	−0.6620	−0.6576	−0.7026
20	9.7460	9.2713	9.3123	0.7064	0.7002	0.7525
40	19.4220	18.8206	18.9384	1.3556	1.3562	1.4554
60	28.2170	27.7584	28.0222	1.9089	1.9293	2.0654
80	36.0310	35.8938	36.2816	2.3748	2.4206	2.5803
100	43.0000	43.2176	43.8145	2.7725	2.8376	3.02543
$\bar{w}_{LIN}$	0.4650	0.4353	0.4352	0.0350	0.0373	0.0373

Table 3. Central displacement and stresses versus load parameter of square clamped (CC) cross ply 0/90 plate with $a / h = 10$.

$\bar{q}$	$\bar{\sigma}_{XX}$		$\bar{\sigma}_{YY}$	
	Thankam et al. 2003	Present	Thankam et al. 2003	Present
−100	−14.2100	−16.2177	−14.21	−15.1525
−80	−13.3280	−14.4082	−13.28	−14.4082
−60	−11.9700	−13.1740	−11.97	−13.1740
−40	−9.8840	−10.1196	−9.884	−10.1196
−20	−6.3090	−5.7065	−6.309	−5.7065
20	7.7410	6.7321	7.741	6.7321
40	14.1160	13.1960	14.116	13.1960
60	19.1260	19.0340	19.126	19.0340
80	23.2730	24.1754	23.273	24.1754
100	26.8130	28.7658	26.813	28.7658
$\bar{w}_{LIN}$	0.3840	0.4136	0.3840	0.4136

Table 4. Central displacement and stresses versus load parameter of square simply-supported (SS) angle ply 45/-45 plate with $a/h = 10$.

$\bar{q}$	$\bar{\sigma}_{XX}$		$\bar{\sigma}_{YY}$	
	Thankam et al. 2003	Present	Thankam et al. 2003	Present
−100	−16.9800	−16.6682	−16.9800	−16.6037
−80	−15.3600	−15.0492	−15.3600	−14.9743
−60	−13.1700	−12.8871	−13.1700	−12.8074
−40	−10.1100	−9.8387	−10.1100	−9.7654
−20	−5.7630	−5.5954	−5.7630	−5.5457
20	6.3330	6.1515	6.3330	6.0767
40	12.0960	11.8130	12.0960	11.6517
60	16.9700	16.6419	16.9700	16.3936
80	21.1100	20.7937	21.1100	20.4630
100	24.7100	24.0000	24.7100	24.0042
$\bar{w}_{LIN}$	0.3140	0.3041	0.3140	0.3009

Table 5. Central displacement and stresses versus load parameter of square hinged (HH) angle ply 45/-45 plate with $a/h = 10$.

$\overline{q}$	$\overline{\sigma}_{XX}$		$\overline{\sigma}_{YY}$	
	Thankam et al. 2003	Present	Thankam et al. 2003	Present
−100	−13.5812	−14.6400	−13.4659	−14.6400
−80	−12.2698	−13.1800	−12.1617	−13.1800
−60	−10.4569	−11.1500	−10.3614	−11.1500
−40	−7.9629	−8.3790	−7.8876	−8.3790
−20	−4.4927	−4.6560	−4.4483	−4.6560
20	5.0711	5.1570	5.0145	5.1570
40	10.0477	10.2150	9.9274	10.2150
60	14.5717	14.8220	14.3851	14.8220
80	18.5663	18.9130	18.3136	18.9130
100	22.1323	22.5590	21.8148	22.5590
$\overline{w}_{LIN}$	0.2510	0.2423	0.2510	0.2423

Table 6. Central displacement and stresses versus load parameter of square clamped (CC) angle ply 45/-45 plate with $a/h = 10$.

A 2x2 quarter plate and 4x4 full plate laminated GLPT models are analyzed and compared with full 8x8 plate FSDT models (Thankam et al. 2003). The results for linear and nonlinear deflections and in plane stresses are presented in Figs. 4,5,6,7 and tables 1-6. It is shown that proposed GLPT model closely agree with FSDT model form literature. Also, the discrepancy between linear and nonlinear solutions are larger for flexible plates, which are the plates with simply supported boundary conditions (SS, SS1), compared to hinged (HH) and clamped (CC) boundary conditions. The study has verified that the change in the sign of the load gives unsymmetrical stress field and symmetrical displacement field, due to non-coincidence of the neutral plane and the mid-plane in laminated composite plates.

Example 4.4. A nonlinear bending of square simply supported (SS1) general quasi-isotropic $(0/45/-45/90)_s$, laminated plate with $a = b = 1$ and $h = 0.1$, made of material:

$$E_1/E_2 = 40,\ G_{12}/E_2 = 0.6,\ G_{13}/E_2 = 0.6,\ G_{23}/E_2 = 0.5,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{31}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\overline{P} = q_0 \cdot a^4/(E_2 h^4)$, the incremental load vector is chosen to be:

$$\{\Delta q\} = \{50, 50, 50, 50, 50\} \cdot \overline{P} \tag{32}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}8$.

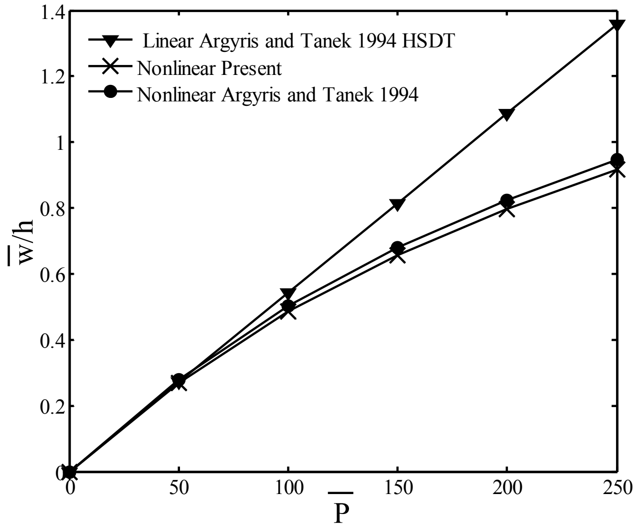

Fig. 8. Nonlinear bending of square simply supported (SS1) general quasi-isotropic (0/45/-45/90)s laminated plate with $a/h=10$; central displacement versus load parameter

A 2x2 quarter plate continuum GLPT model is compared with 8x8 full plate HSDT model (Argyris and Tanek 1994). The results for linear and nonlinear deflections are presented in Fig. 8. It is shown that proposed GLPT model closely agree with HSDT model form literature, with the faster convergence.

5. Conclusion

In this paper a laminated layerwise finite element model for geometrically nonlinear small strain, large deflection analysis of laminated composite plates is derived using the PVD. The accuracy of the model is verified calculating nonlinear response of plates with different mechanical properties, which are isotropic, orthotropic and anisotropic (cross ply and angle ply), different plate thickness, different boundary conditions and different sign of the load (unloading/loading). In despite of its mathematical complexity, proposed model has shown better convergence characteristics than ESL models of CLPT, FDST and HSDT, still with less computational cost than 3D elasticity model. Moreover, present model has no shear locking problems, compared to ESL models, or aspect ratio problems, as the 3D finite element may have when analyzing thin plate behavior. The analysis has also shown that the discrepancy of nonlinear from linear response is greater for flexible plates, such as thick compared to thin plates, or plates with SS compared to hinged (HH) and clamped (CC) boundary conditions. It is verified that the change of the sign of load (unloading/loading) has no influence on displacement field, while the stress field dependents on the sign of the load, due to non coincidence of the natural plane and the mid plane of the plate.

Геметријски нелинеарна анализа ламинираних композитних плоца корисцењем слојевитог модела померања

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У овом раду је развијен геометријски нелинеаран ламинирани модел коначних елемената коришћењем принципа виртуалних померања. 3Д једначине еластичности су сведене на 2Д проблем коришћењем кинематских претпоставки базираних на претпостављеном пољу померања слојева Редија. Са претпостављеним пољем померања, нелинеарним Грин – Лагранжевим релацијама за мале деформације и велика померања, и линеарним ортотропним материјалним карактеристикама за сваки слој (плочу), принцип виртуалних померања је коришћен за добијање слабе форме. Слаба форма или нелинеарне интегралне равнотежне једначине дискретизовани су коришћењем изопараметарских апроксимација у коначним елементима. Нелинеарне инкременталне алгебарске једначине су решаване поступком директних итеграција. Написан је оригинални МАТЛАБ програм за решавање методом коначних елемената, који је коришћен је за истраживање геометријски нелинеарних ефеката на поља померања и напона код танкии и дебелих, изотропних, ортотропних и анизотропних слојевитих композитних плоча са променљивим граничним условима и знаком оптерећења (оптерећење - растерећење). Тачност нумеричког модела је верификована упоређивањем са резултатима из литературе и линеарним решењима из претходног рада. Изведени су одговарајући закључци.

Кључне речи: геометријски нелинеарна анализа, слојевити модел коначних елемента

References

Argyris, J. and Tanek, L. (1994), “Linear and geometrically nonlinear bending of isotropic and multilayered composite plates by the natural mode method”, Computer methods in applied mechanics and engineering, 113, 207-251.

Ćetković M, Vuksanović Dj (2009), “Bending, Free Vibrations and Buckling of Laminated Composite and Sandwich Plates Using a Layerwise Displacement Model, Composite Structures, 88(2) pp. 219-227

Ćetković, M. (2005), Application of finite element method on generalized laminated plate theory, Master Thesis, in serbian, Faculty of Civil Engineering in Belgrade, Serbia.

Ćetković, M. (2011), Nonlinear behaviour of laminated composite plates, PhD Thesis, in serbian, Faculty of Civil Engineering in Belgrade, Serbia.

Polat, C. and Ulucan, Z. (2007), “Geometrically non-linear analysis of axisymmetric plates and shells”, International Journal of Science and Technology, 2(1), pp. 33-40.

Reddy JN, Barbero EJ, Teply JL. (1989), A plate bending element based on a generalized laminated plate theory, International Journal for Numerical Methods in Engineering, 28, pp. 2275-2292

Reddy, J.N. (2004), Mechanics of laminated composite plates-theory and analysis, CRC press.

Reddy, J.N. and Chao, W.C. (1983), "Nonlinear bending of bimodular-material plates", International Journal of Solids and Structures, 19(3), pp. 229-237.

Tanriover, H. and Senocak, E. (2004), "Large deflection analysis of unsymmetrically laminated composite plates: analytical-numerical type approach", International Journal of Non-linear Mechanics, 39, pp. 1385-1392.

Thankam, V.S. and Singh, G. and Rao, G.V. and Rath, A.K. (2003), "Shear flexible element based on coupled displacement field for large deflection analysis of laminated plates", Computers and Structures, 81, 309-320.

Vuksanović Dj. (2000),"Linear analysis of laminated composite plates using single layer higher-order discrete models", Composite Structures, 48, pp. 205-211.

Zhang, Y. and Wang, S. and Peterson, B. (2003) "Large deflection analysis of composite laminates", Journal of Materials Processing Technology, 138, pp. 34-40.

Zhang, Y.X. and Kim, K.S. (2006), "Geometrically nonlinear analysis of laminated composite plates by two new displacement-based quadrilateral plate elements", Composite Structures, 72, pp. 301-310.