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Large Deflection of Deep Beams on Elastic Foundations

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Abstract

This research deals with the geometric nonlinear behavior of beams resting on Winkler foundation. Timoshenko's deep beam theory is extended to include the effect of large deflection theory. The finite difference method was used to solve the problem deep beams and the obtained results were compared. An incremental load approach with Newton-Raphson iteration computational technique was used for solving the nonlinear sets of node equilibrium equations in the finite difference method.

In the finite element method (ANSYS program), the element SHELL 43 incorporated in ANSYS 5.4 was used. The element has four nodes with six degrees of freedom at each node: translations in the nodal x, y, and z- directions and rotations about the nodal x, y, and z- directions.

Several important parameters were incorporated in the analysis to study the effects of vertical subgrade reaction, beam width, and beam depth to length ratio on the deflections, bending moments and shear forces. The results obtained from this method were compared with exact and numerical methods to check the accuracy of the solutions. Good agreements were found, the maximum difference in deflection at midspan from the finite elements and the finite differences was (0.79%). Also, the difference between the exact solution and the present finite difference was found to be (0.86%).

Keywords: Deep beams, elastic foundations, finite differences, finite elements, large deflections.

1 Introduction

The modifications to the classical theories of thin or shallow beams (Euler-Bernoulli) have started with the work of Timoshenko in 1921 on vibration problems of prismatic bars. The normal lines to the middle plane are allowed to rotate independently of rotations (or slopes) of the middle plane. Thus, transverse shearing deformations are considered. This improvement to the classical beam theory has helped to solve problems of deep beams by the approaches of strength of materials.

Winkler (1867) proposed the first model of beam on an elastic foundation based on pure bending beam theory, later Pasternak in 1954 proposed the shear model based on the assumption of pure shear of the beam. Both of these two models take an extreme point of view on the deformation behavior of the beam. Timoshenko in 1921 proposed models for the vibration of deep beams; his beam theory still attracts people's attention for studying the static and dynamic responses of deep beams (Bowles 1974).

Biot (1937) considered the problem of bending, under a concentrated load, of infinite flexible beams on a homogeneous elastic-isotopic subgrade.

Levinson (1949) suggested that the contact pressure is represented by a number of redundant reactions which would create a set of simultaneous equations in terms of pressures diagram coordinates and elasticity constants.

Terzaghi (1955) established a number of equations to calculate the modulus of subgrade reaction for cohesive and cohesionless soils, depending on plate load test results.

Cheung and Nag (1968) studied the effects of separation of contact surfaces due to uplift forces. In addition, they have enabled the prediction of the bending and torsion moments in the plate sections by adopting three degrees of freedom per node.

Bowles (1974) developed a computer program to carry out the analysis of beams on elastic foundation by using the finite element method, in which Winkler model is adopted to represent the elastic foundation.

Selvadurai (1979) presented a theoretical analysis of the interaction between a rigid circular foundation resting on elastic half space and a distributed external load of finite extent which acts at an exterior region of the half space.

Yankelevsky et al. (1988) presented an iterative procedure for the analysis of beams resting on nonlinear elastic foundation based on the exact solution for beams on a linear elastic foundation.

Yamaguchi et al. (1999) presented a finite element formulation for the large displacement analysis of beams. It was based on the degeneration approach.

Yin (2000) derived the governing ordinary differential equation for a reinforced Timoshenko beam on an elastic foundation.

Guo and Wietsman (2002) made an analytical method, accompanied by a numerical scheme, to evaluate the response of beams on the space-varying foundation modulus, which is called the modulus of subgrade reaction ($K_z=K_z(x)$).

The geometric nonlinear effects of beams could become significant even at relatively small geometry changes. In more advanced nonlinear analysis there are several significant effects such as:

1. Change of member lateral stiffness (i.e. stability problem).

2. Change in member length due to bowing effects. These are caused by the interaction between the bending moment and the axial force in the members.

3. Shear deformation and connection nonlinearity.

4. Excessive or large displacement, when the displacements become large enough to cause significant changes in the geometry of the structure, the equilibrium equations must be formulated for the updated configuration.

The aim of the present study is to analyze deep beams by using the finite difference and the finite element methods with large deflections. The beam is resting on an elastic foundation with Winkler compressional resistance, and loaded generally (by both transverse distributed load and distributed moment), as the effect of transverse shearing deformations is included. The governing differential equations of deep beams (in terms of deflection w and section rotation ψ) are derived and then converted into finite differences.

A computer program (LDBEF) (Large Deflections of Beams on Elastic Foundations) (in FORTRAN language) was formed. This program solves the problem by converting the differential equations into finite differences. Then by assembling the finite difference equations to obtain a system of simultaneous algebraic equations, the equations are solved by using Gauss- Jordan method. The deflections and rotations at each node are obtained. The deflection for each step is compared with the value of the previous step and the iteration proceeds until convergence is obtained. The shearing forces and bending moments are obtained by simple substitutions of the obtained deflections and rotations into the finite difference equations of moment and shear. The obtained solution is compared with available exact and numerical results to assess the accuracy of the method used in this study.

2 Assumptions and governing equations for deep beams

The main assumptions are:

1) Plane cross sections before bending remain plane after bending.

2) The cross section will have additional rotation due to transverse shear. Warping of the cross section by transverse shear will be taken into consideration by introducing a shear correction factor (c^2).

The governing equations of deep beams on elastic foundations characterized by Winkler model for compressional resistance (Fig.1) could be obtained:

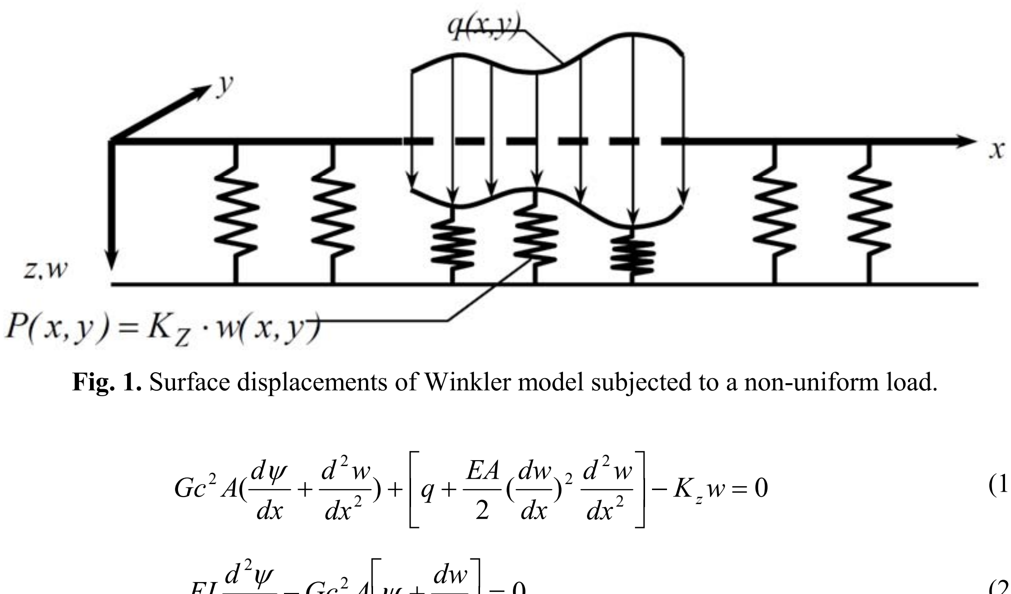

Fig. 1. Surface displacements of Winkler model subjected to a non-uniform load.

$$Gc^2 A(\frac{d\psi}{dx} + \frac{d^2 w}{dx^2}) + \left[q + \frac{EA}{2}(\frac{dw}{dx})^2 \frac{d^2 w}{dx^2} \right] - K_z w = 0 \tag{1}$$

$$EI\frac{d^2\psi}{dx^2} - Gc^2 A\left[\psi + \frac{dw}{dx} \right] = 0 \tag{2}$$

where G is the shear modulus , c^2 is the shear correction factor (c^2 =5/6 for rectangular cross sections and c^2=1 for I-sections), A is the cross-sectional area of the beam, ψ is the rotation of the transverse sections in *xz*-plane of the beam, w is the transverse deflection, E is the modulus of elasticity of the beam material, q is the transverse load per unit length, K_z is the modulus of subgrade reaction in z-direction and I is the moment of inertia of the beam section. In case of the depth of the beam decreases (thin beam) the shear modulus becomes infinite and equation 2 vanishes as $\psi = -\frac{dw}{dx}$ and equation 1 reduces to the case of large deflection of thin beams [Theeban (2007)].

3 Finite difference method

The finite difference method is one of the most general numerical techniques. In applying this method, the derivatives in the governing differential equations under consideration are replaced by differences at selected points. These points or nodes are making the finite difference mesh. In the analysis of deep beams by this method, the coupled differential equations at each point (or node) are replaced by coupled difference equations. By assembling the coupled difference equations for all nodes, a number of simultaneous algebraic equations are obtained and solved by Gauss-Jordan method.

The beam is divided into intervals of (Δx) in the (x) direction as shown in figure 2, assuming (n) to represent the number of nodes and (i) the node number under consideration. In the finite difference method, the curve profile of the beam deflection is approximated by a straight line between nodes for the finite difference expressions of the first derivatives and by a parabola for the second derivatives.

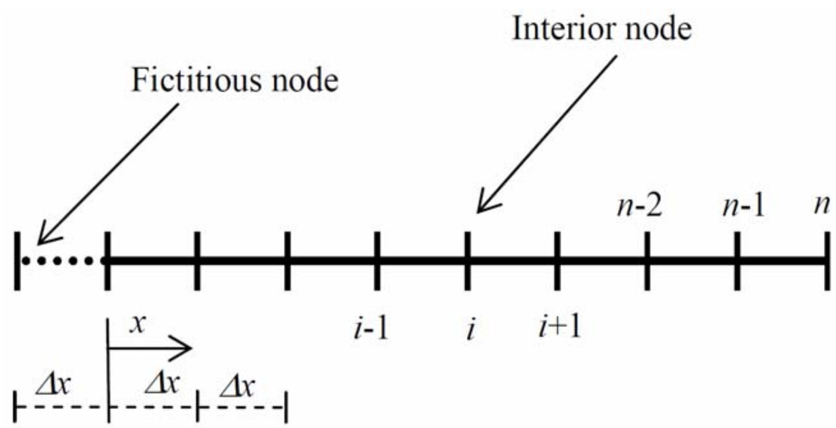

Fig. 2. Finite difference mesh for deep beam.

The central differences for the first and second derivatives of the transverse deflection and rotation at node (i) are:

$$\left(\frac{d\Phi}{dx}\right)_i = \left(\frac{\Phi_{i+1} - \Phi_{i-1}}{2\Delta x}\right) \tag{3}$$

$$\left(\frac{d^2\Phi}{d x^2}\right)_i = \left(\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\Delta x^2}\right) \tag{4}$$

where Φ is *w* or ψ.

To get a solution for the deflection and rotation, only one fictitious point is needed to define the deflection and rotation beyond the boundary.

Using above equations the following finite difference equations are produced for an interior node (*i*):

$$Gc^2A[\frac{\psi_{i+1}-\psi_{i-1}}{2\Delta x}+(\frac{w_{i+1}-2w_i+w_{i-1}}{(\Delta x)^2})]+[q_i+ \frac{EA}{2}(\frac{w_{i+1}-w_{i-1}}{2\Delta x})^2(\frac{w_{i+1}-2w_i+w_{i-1}}{(\Delta x)^2})]-K_z.w_i=0 \tag{5}$$

$$Gc^2A[\frac{\psi_{i+1}-\psi_{i-1}}{2\Delta x}]+\frac{Gc^2A}{\Delta x^2}[w_{i+1}-2w_i-\frac{K_z w_i \Delta x^2}{Gc^2A}+w_{i-1}] +[q_i+\frac{EA}{2}(\frac{w_{i+1}-w_{i-1}}{2\Delta x})^2(\frac{w_{i+1}-2w_i+w_{i-1}}{(\Delta x)^2})]=0 \tag{6}$$

The solution of the governing differential equations of deep beams must simultaneously satisfy the differential equations and the boundary conditions for any given beam problem.

Boundary conditions are represented in finite difference form by replacing the derivatives in the mathematical expressions of various boundary conditions by their finite difference approximations. When central differences are used at the boundary nodes, fictitious points outside the beam are required. These may be defined in terms of the inside points when the behavior of the beam functions are known at the boundary nodes.

3.1 Simply supported edge

The moment value at the boundary node is zero, from which the rotation ψ at the fictitious points can be related to the rotation at the interior nodes as follows:

$$M=0 \quad = EI(\frac{d\psi}{dx}) \tag{7}$$

$$\frac{d\psi}{dx}=0$$

thus,

$$(\frac{d\psi}{dx})_{boundary}=(\frac{\psi_{interior}-\psi_{fictitious}}{2\Delta\ x})\Rightarrow\psi_{interior}=\psi_{fictitious} \tag{8}$$

Since the second derivative for deflection $(\frac{d^2w}{dx^2})$ at the boundary node is unknown, therefore the deflection *w* at the fictitious point cannot be related to the deflections at interior nodes. Thus, a forward difference or a backward difference is used to define the deflections at the interior nodes near the boundaries:

$$(\frac{dw}{dx})_i = (\frac{w_{i+1} - w_i}{\Delta x}) \quad \text{(Forward)} \tag{9}$$

$$(\frac{dw}{dx})_i = (\frac{w_i - w_{i-1}}{\Delta x}) \quad \text{(Backward)} \tag{10}$$

Knowing that deflection is zero at the boundary nodes, only equation (6) is used at the boundary nodes, since this equation has only the first derivative for deflection.

3.2 Clamped edge

Since the second derivative is the maximum derivative in the governing equations for deep beams, therefore only one point beyond the node under consideration is needed if the central difference is used. The deflection and rotation are zero at the boundary nodes. Therefore there will be no need for a fictitious point beyond the boundary if the first interior node is considered as the first point under consideration.

3.3 Free edge

The moment and shear force values at the boundary node are zero. Since the second derivatives for deflection and rotation at the boundary nodes are unknown, the deflection w and rotation ψ at the fictitious points cannot be related to the deflection and rotation at interior nodes. Therefore equations (5.5) and (5.8) are not useful in this case. Thus, shear force and moment equations will be used as shown below:

$$\frac{M}{EI} = \frac{\psi_{i+1} - \psi_i}{\Delta x} \tag{11}$$

$$\frac{Q}{Gc^2 A} = [\psi_i + (\frac{w_{i+1} - w_i}{\Delta x})] \tag{12}$$

4 Applications

The cantilever under a concentrated load shown in figure 3 which was solved by Yamaguchi et al. in (1999) is considered. The results of deflections are shown in figure 4 and Table 1 to check the accuracy of the developed program. Figure 4 and Table 1 indicate that the results are in excellent agreement (percentage difference 0.4% with other studies). The number of nodes used in the finite differences is 30, and the number of elements used in the finite elements (ANSYS program) is 40. The finite element mesh is shown in figure 5.

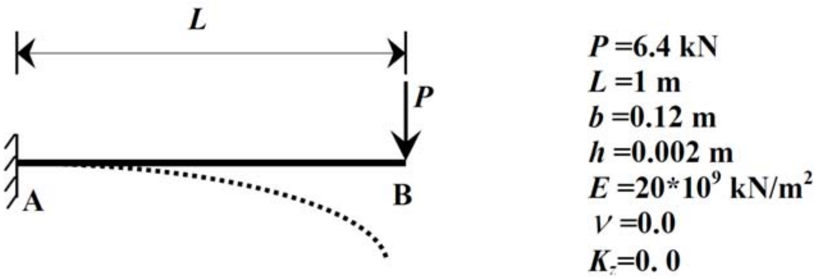

Fig. 3. Cantilever under a concentrated load.

Present Study max. deflection (mm)	Yamaguchi et al. 1999 max. deflection (mm)	ANSYS program max. deflection (mm)
66.733	66.996	66.966

Table 1. Comparison of maximum deflection with other methods for the cantilever beam under a concentrated load.

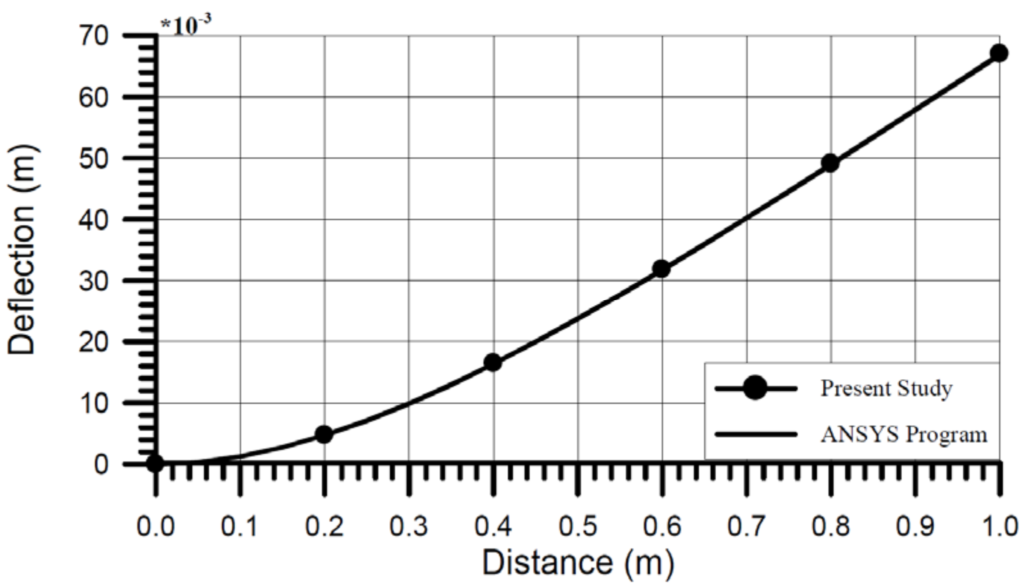

Fig. 4. Deflection curves for cantilever under a concentrated load.

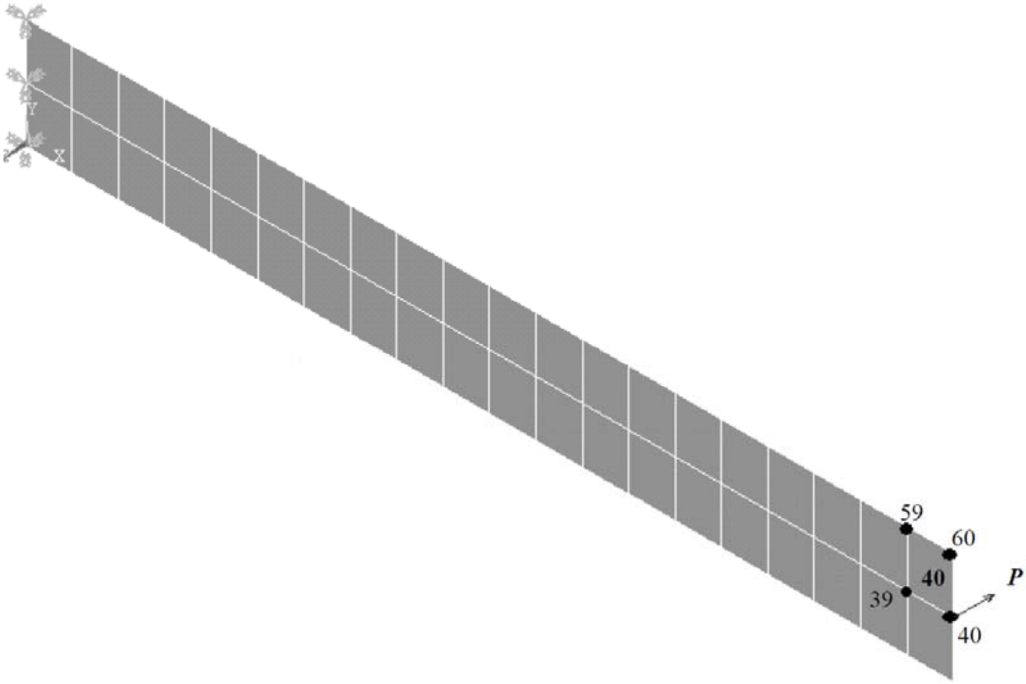

Fig. 5. Finite element mesh used for the cantilever under a concentrated load (ANSYS program).

5 Parametric study

A parametric study is performed to investigate the influence of several important parameters on the behavior of beams resting on Winkler foundation and subjected to static loads. To study the effects of these parameters, a simply supported deep beam will be considered in this study, the geometric quantities, material constants and the load values are given in figure 6.

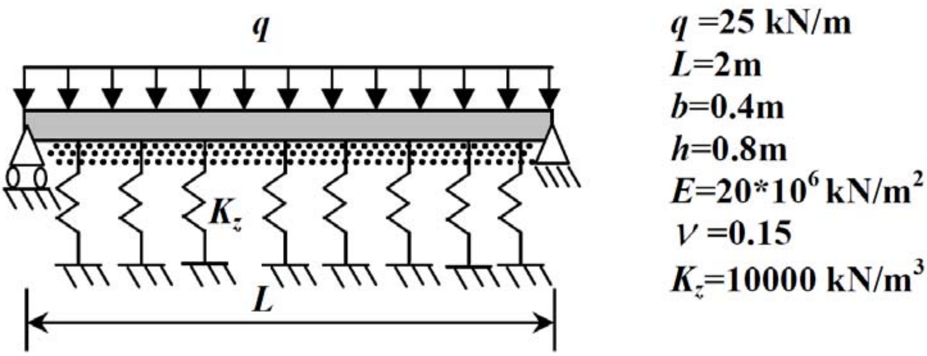

Fig. 6. Uniformly loaded simply supported deep beam resting on Winkler foundation.

5.1 Effect of depth to span ratio (h/L)

To show the effect of (*h/L*) ratio, different values of depth to span ratio are considered. The values are (0.2, 0.3, and 0.4 to 1). Figure 7 shows the effect of variation of (*h/L*) on the maximum deflection (at midspan) for a simply supported beam under a uniformly distributed load. From this figure, the maximum deflection will decrease at a decreasing rate as the beam depth is increased. It was found that by increasing the ratio (*h/L*) from (0.2 to 1); the maximum deflection for a simply supported beam under a uniformly distributed load is decreased by (97.4%).

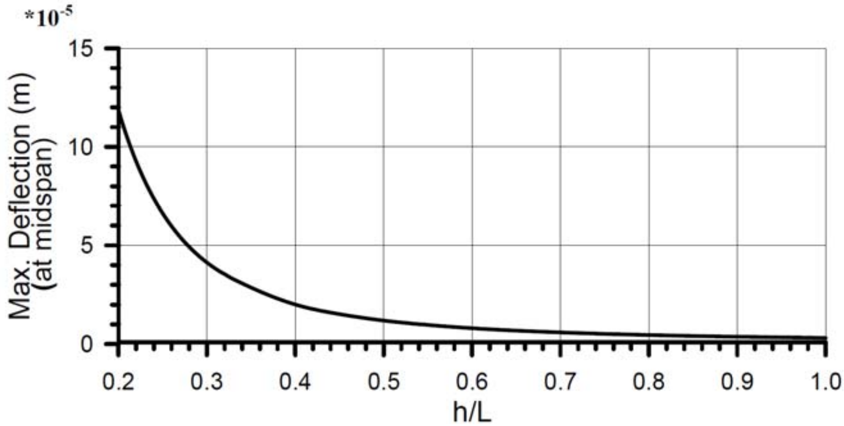

Fig. 7. Effect of (*h/L*) on maximum deflection (at midspan) for a simply supported beam under a uniformly distributed load.

Figure 8 shows the effect of variation of (*h/L*) on maximum moment for a simply supported beam under a uniformly distributed load. From this figure, the maximum moment will increase at a decreasing rate as the beam depth is increased. It was found that by increasing the ratio (*h/L*) from (0.2 to 1.0); the maximum moment for the simply supported beam under a uniformly distributed load is increased by (9.7%).

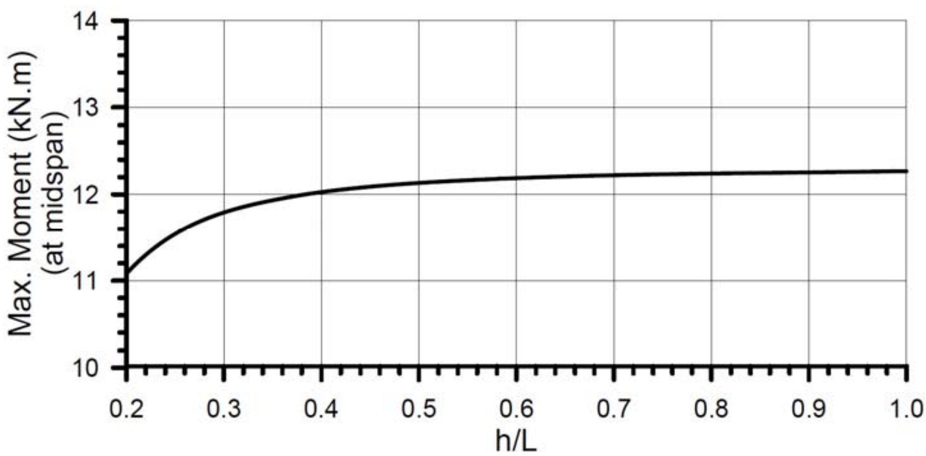

Fig. 8. Effect of (*h/L*) on maximum moment for a simply supported beam under a uniformly distributed load.

Figure 9 shows the variation of (*h/L*) with maximum shear force for a simply supported beam under a uniformly distributed load. From this figure, the maximum shear force will increase at a decreasing rate as the beam depth is increased. It was found that by increasing the ratio (*h/L*) from (0.2 to 1.0); the maximum shear force for the simply supported beam under a uniformly distributed load is increased by (9.1%).

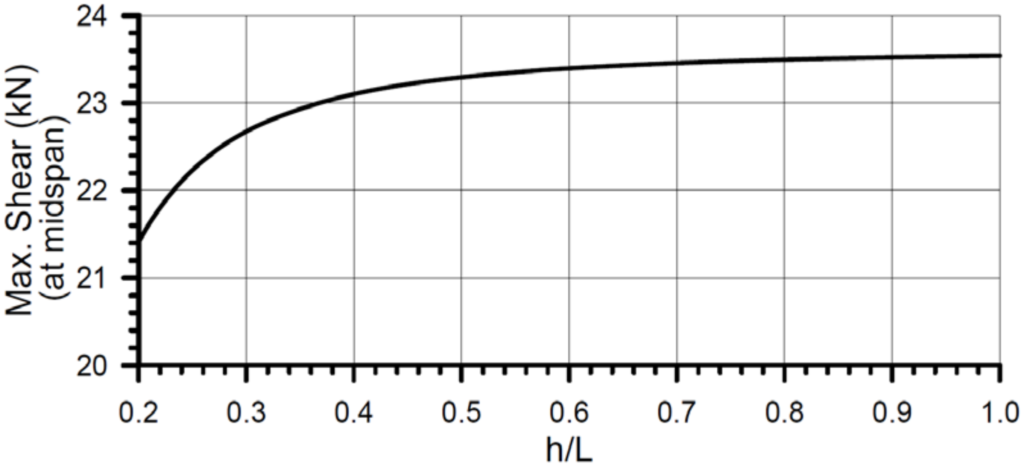

Fig. 9. Effect of (*h/L*) on maximum shear force for a simply supported beam under a uniformly distributed load.

5.2 Effect of vertical subgrade reaction (K_z)

To show the effect of vertical subgrade reaction (K_z), different values of vertical subgrade reaction (K_z) are considered. The values are (0.0, 10000, and 30000 to 60000 kN/m^3). Figure 10 shows the effect of variation of vertical subgrade reactions on maximum deflection for a simply supported deep beam under a uniformly distributed load. From this figure, the maximum deflection will decrease at a decreasing rate as the vertical subgrade reaction is increased. It was found that by increasing the vertical subgrade reaction from (0.0 to 60000 kN/m^3); the maximum deflection for the simply supported deep beam under a uniformly distributed load is decreased by (3.77%).

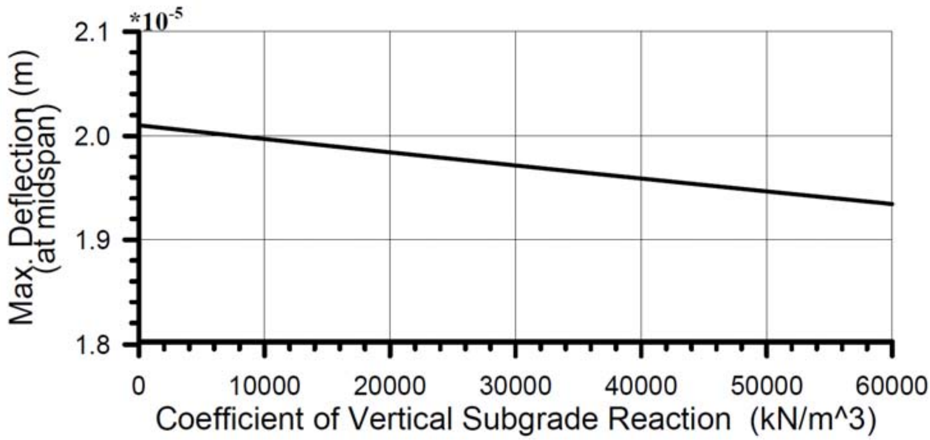

Fig. 10. Effect of vertical subgrade reaction on maximum deflection (at midspan) for a simply supported deep beam under a uniformly distributed load.

Figure 11 shows the effect of variation of vertical subgrade reactions on maximum moment of a simply supported deep beam under a uniformly distributed load. From this figure, the maximum moment will decrease at a decreasing rate as the vertical subgrade reaction is increased. It was found that by increasing the vertical subgrade reaction from (0.0 to 60000 kN/m^3); the maximum moment for the simply supported deep beam under a uniformly distributed load is decreased by (3.8%).

Figure 12 shows the effect of variation of vertical subgrade reaction on maximum shear force of a simply supported deep beam under a uniformly distributed load. From this figure, the maximum shear force will decrease at a decreasing rate as the vertical subgrade reaction is increased. It was found that by increasing the vertical subgrade reaction from (0.0 to 60000 kN/m^3); the maximum shear force for the simply supported deep beam under a uniformly distributed load is decreased by (3.2%).

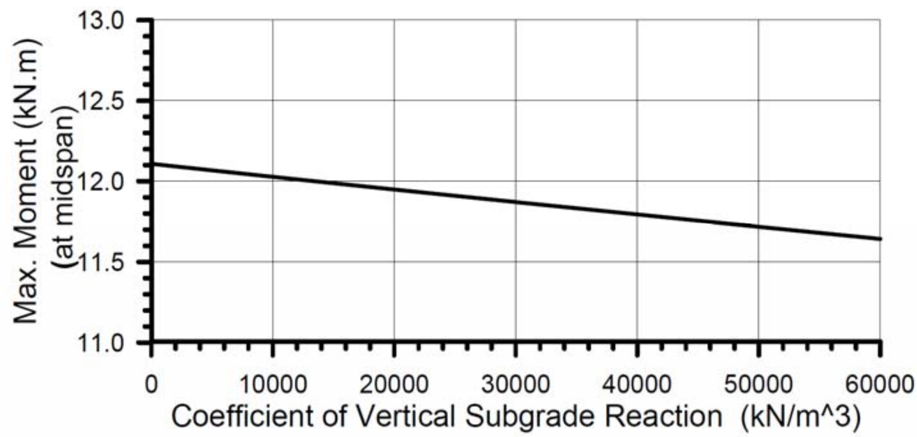

Fig. 11. Effect of vertical subgrade reaction on maximum moment for a simply supported deep beam under a uniformly distributed load.

5. 3 Effect of beam width (b)

To show the effect of beam width on the results of deep beams, various values of width are considered. The values are (0.2, 0.3 to 0.6 m). Figure 13 shows the effect of variation of beam width on maximum deflection for a simply supported deep beam under a uniformly distributed load. From this figure, the maximum deflection will decrease at a decreasing rate as the beam width is increased. It was found that by increasing the beam width from (0.2 to 0.6 m), the maximum deflection for the simply supported deep beam under a uniformly distributed load is decreased by (69.1%).

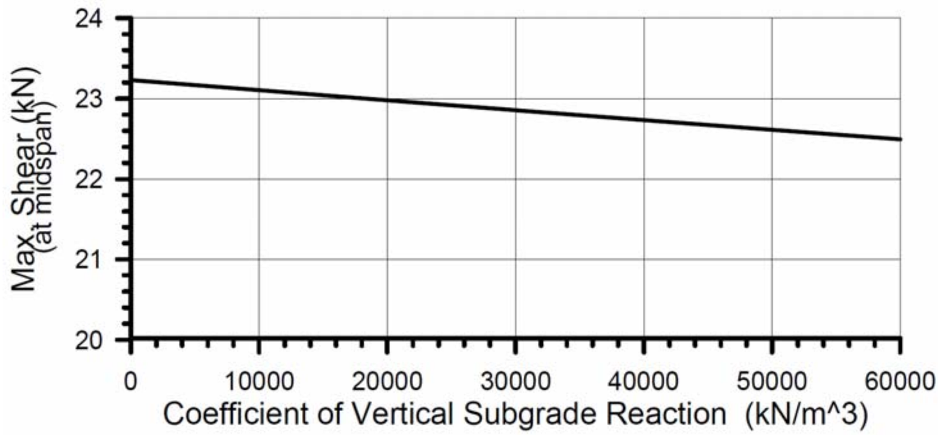

Fig. 12. Effect of vertical subgrade reaction on maximum shear force for a simply supported deep beam under a uniformly distributed load.

Figure 14 shows the effect of variation of beam width on maximum moment of a simply supported deep beam under a uniformly distributed load. From this figure, the maximum moment will increase at a decreasing rate as the beam width is increased. It was found that by increasing the beam width from (0.2 to 0.6 m), the maximum moment for the simply supported deep beam under a uniformly distributed load is increased by (0.9%). Figure 15 shows the effect of variation of beam width on maximum shear force of a simply supported deep beam under a uniformly distributed load. From this figure, the maximum shear force will increase at a decreasing rate as the beam width is increased. It was found that by increasing the beam width from (0.2 to 0.6 m), the maximum shear force for the simply supported deep beam under a uniformly distributed load is increased by (0.7%).

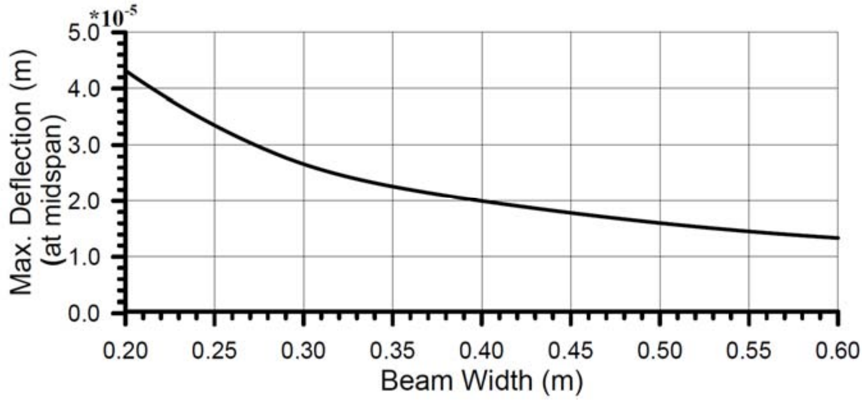

Fig. 13. Effect of beam width on maximum deflection (at midspan) for a simply supported deep beam under a uniformly distributed load.

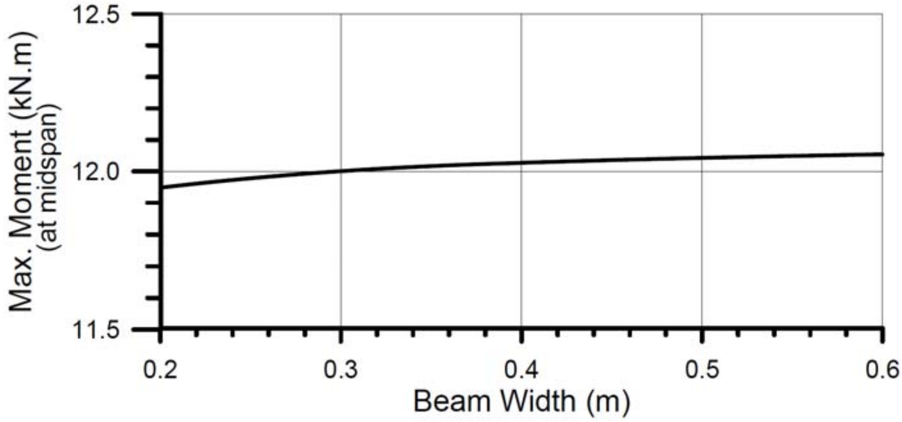

Fig. 14. Effect of beam width on maximum moment for a simply supported deep beam under a uniformly distributed load.

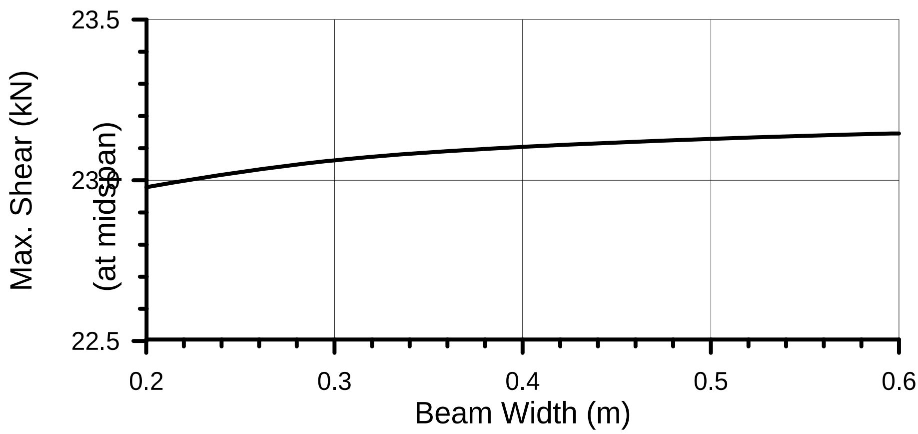

Fig. 15. Effect of beam width on maximum shear force for a simply supported deep beam under a uniformly distributed load.

6. Conclusions

From this study, the main conclusions are given below:

- The results obtained from this method and the results obtained from other researches are plotted together to check the accuracy of the used finite difference method. Good agreements are obtained by using this method; the maximum difference in maximum deflection between the results from finite elements and from finite differences is (0.79%). Also, the difference between the exact solution and the finite difference is found to be (0.86%).
- The effect of increasing beam depth on the deflections is found to be more significant than the effect on the stress resultants, (bending moments and shearing forces). In simply supported beam cases under a full uniform load, when (*h/L*) varied from (0.2 to 1.0), the percentages are (97.4%), (9.7%), and (9.1%) respectively.
- The effect of varying the modulus of elastic foundations on the deflections and internal stress resultants of deep beams becomes insignificant as the depth increases.
- The effect of varying the beam width on the deflection and internal stress resultants of deep beams becomes insignificant as the depth increases.

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