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# Modeling of One-Dimensional Unsteady Open Channel Flows in Interaction with Reservoirs, Dams and Hydropower Plant Objects

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## Abstract

A number of existing software solutions are used to model one-dimensional unsteady open-channel flow using the standard boundary and initial conditions. Internal boundary conditions are two flow trajectories, related by special conditions and each is associated with two equations describing the water flow. The special conditions include the matrix formulation of limitations that apply to the flow inside the two connected trajectories. Branching, pipe joints and flow over the weir are some of the modeling tasks usually solved. This paper presents the algorithms used for the solution of the complex case of hydropower plant objects as internal boundary cases. The complexity of the problem lies in the fact that the hydropower plant is managed according to the electricity generation demand which is dependent both upon the upstream and downstream water levels, as well as the discharge. Since all state values are coupled and determined as system solutions, the algorithm leads to an iterative procedure of solving a system of non-linear equations. Software used in solving the standard, complete, dynamic equations of flow in one-dimensional unsteady open channel flow and through the control structures, was adopted as the foundation applied in simulation of the "Iron Gate" ("Đerdap" in Serbian; "Portile de Fier" in Romanian) system, with detailed modeling of three objects of the hydropower plants "Iron Gate 1", "Iron Gate 2" and "Gogoš".

**Keywords:** Flow, simulation, updating, open channel flow, hydropower plant, internal boundary conditions

## 1. Introduction

Hydropower systems play an important role in the integrated electricity generation and transmission system due to low expenses of utilization and high flexibility that allows for high-quality management of the in real-time (Divac et al. 2009). Besides, their quality is becoming apparent with the growth of interest in environmental issues and electricity generation issues, as well as the support to other systems based on renewable energy sources (wind-driven generators, solar energy and similar). Negative effects of exploitation of hydropower potential are usually related to the impact of storage construction and newly created conditions in the environment, and far less to the actual utilization of resources. These effects can be comprehended through changes in natural environment, population displacement and potential accidents.

Electricity generation in hydropower plants is directly conditioned by the transformation of rainfall into runoff in the catchment area; hence, electricity generation is often affected by the dry seasons or snowmelt periods and similar extreme events. Electricity generation, in long-term, can be also affected by the climate conditions. Exploitation of hydropower potential is conditioned by numerous limitations related to multi-purpose use of water resources (Labadie 2004). At the same time, the main system input is not manageable since it is being dictated by the natural hydrologic processes and inflow forecast is reliable only to a certain extent.

The optimum management of hydropower systems includes the simulation of physical phenomena in the storage and operation of electricity generation objects. Depending on the size of physical objects and the required level of simulation accuracy, various hydraulic models are applied. These can be of purely balance models and, but also can be numerically complex models that implement the full system of equations of flow in open channels and storages. As in the case of hydraulic flow models, electricity generation objects can be presented in several ways, depending on the data availability and the required level of accuracy.

The most common approach to solving the problem of unsteady open channel flow is the finite difference method. The method performs the approximation of Saint-Venant equations in space and time. Depending on the effects monitored by the model simulation, certain equation terms may be approximately defined by the first, second or higher order of approximation. The choice of the approximation order may impact the model applicability. The most commonly used models are 1D/2D SOBEK model by the company Delft Hydraulics, Mike11 (1D flow) and Mike21 (2D flow) by the Danish Hydraulics Institute, ISIS model by the Wallingford Software and, finally, HEC-RAS created by the US Army Corps of Engineers.

All these models may take into account the impact of objects located in the flow, such as dams, gates and similar. It is assumed that the dimensions of these objects in the flow direction are negligible relative to the flow width; hence, they are treated as points. These points represent the places of discontinuity in terms of discharge and energy; hence Saint-Venant equations are not applicable there. In these places an additional connection between the headwater (upstream water level), discharge through the object and tailwater (downstream water level) should be formulated. It is considered that the point where headwater level is calculated should be as close to the object as possible, provided that the vertical component of acceleration can be neglected. Also, tailwater level is calculated in the closest possible downstream cross-section, where the flow can be considered approximately horizontal. Besides that, water retention in object's surroundings can also be neglected.

Present paper shows the mathematical model of unsteady flow developed for the needs of the complex software environment for support to the management of hydropower system “Iron Gate” as a whole. The management is performed to meet the requirements of Serbian and Romanian electricity generation and transmission systems, which differ in terms of energy requests and time zones, and to meet the series of limitations related to the limitations on control profiles on River Danube, which were defined by the state-level documents. A complex numerical model, based on the finite difference method, has been developed in order to meet the requirements of the calculations related to hydropower and management of system exploitation. It also meets the requirements related to its application in efficient operational decision-making, as well as in designing of the studies based on calculations in relation to hydraulics and energy.

## 2. Theoretical background of unsteady open channel flows

In the analysis of unsteady flows (Mahmood et al., 1975) it is necessary to define the calculation elements and introduce algebraic approximations in order to transform the basic flow equations into two equations wherein the values of water level and discharge at the end of elements are used. Since geometrical characteristics of the riverbed are in real problems most often highly complex, it is practically impossible to solve the basic model equations analytically. This is the reason why the entire flow section used in the calculation is divided into numerous elements, on which certain approximations can be applied and, while, at the same time, the desired calculation accuracy can still be preserved. The basic equations of unsteady flow directly relate the values of discharge and water level at the end of each element. Each two adjacent elements are connected by a common node; hence, the equations of all elements constitute a system of equation that describes the flow in the whole system. The results of solving this system of equations are the values of discharge and water level in all nodes. Since this is the calculation of unsteady flow, the equations include certain time-dependent terms. Due to time-dependency the values of discharge and water level have to be determined for each time section. Time axis is divided into intervals which must be sufficiently short for the approximation of the basic algebraic equations to be sufficiently correct.

If the values of discharge and/or water level in certain points (nodes) in the system are known, then these values can be defined as the boundary conditions for solving the equations of the system. The nodes with assigned known values of discharge or water level values are called control points. Also, the control points can be assigned with the known relations between the water level and discharge, sudden discharge changes or interaction between flows, which are not described by basic equations. For example, the control point may represent a dam with a power plant and a spillway, where the dependency of discharge upon the headwater or tailwater level is defined, where the headwater level is equal to the level at the end of the calculation element that directs the water towards the dam and the tailwater level is equal to the level at the start of the calculation element that leads the water away from the dam.

For the calculation of the unsteady flow it is necessary to define initial condition, i.e. to determine the state in the computational section at the initial moment of time. The initial condition is defined by the calculation of the steady flow (i.e. the calculation of the water level line in the regime of steady flow), which defines water levels in all transversal profiles of the calculation section, for the discharge at the initial time of the calculation.

One additional difference between unsteady flow calculation and the steady-flow calculation is the method of definition of boundary conditions (i.e. the data assigned to the boundary profiles of the calculation section). As the number of equations is lower than the number of unknown values, some additional conditions will have to be introduced. For example, one calculation section should be divided into 6 calculation elements, i.e. it will contain 7 nodes. As there are two unknown values in each node, in this case there 14 unknowns in total and only 12 equations (two per calculation element). This is the reason why the conditions applicable to the system boundaries are necessary for determination of the unknown values. In contrast to the steady-flow calculation, when, as a rule, only downstream boundary condition is assigned, for the calculation of unsteady flow in the quiet hydraulic regime it is necessary to assign the values both on the upstream and the downstream boundary of the flow space. These values may be assigned in the one of the three following forms: as the known discharge that is a function of time (hydrograph), as the known water level that is a function of time (level-gram), or as a relation between the discharge and the level (discharge curve). Upstream boundary condition is most often assigned in the form of a hydrograph, and the downstream one in the form of a discharge curve. It is also possible to define the internal boundary conditions, what is done on the profiles within the computational section on the location of flow "interruption" (confluence, dam, lateral spillway etc.). Internal boundary conditions are defined by the means of the equation of mass conservation and the equation of momentum conservation for steady flows. A significant part of the analysis of unsteady flow is the identification and defining of the points where the internal boundary conditions should be assigned.

In the unsteady-flow analysis the three principles of conservation are used – the principle of water mass conservation, the principle of water mechanical energy conservation and the principle of water momentum conservation. Conservation of mass may be treated as conservation of volume if the water density is constant. The equation derived by the application of the principle of mass conservation is often referred to as the "continuity equation". In addition to mass conservation, the mathematical model of flow also uses the principle of momentum conservation and the reason for this is that the use of the continuity equation and momentum conservation equation is simpler than the use of the principle of mechanical energy conservation.

### *2.1. Equation of one-dimensional unsteady flow*

The integral form of the equation of one-dimensional unsteady free-surface flow represents the description of the principles of mass and momentum conservation for a certain control volume. Figure 1 shows the control volume with the upstream and downstream sides positioned perpendicularly to the water flow. Other boundaries are defined by the riverbed contours, while the upper side of the flow space is determined by the free surface.

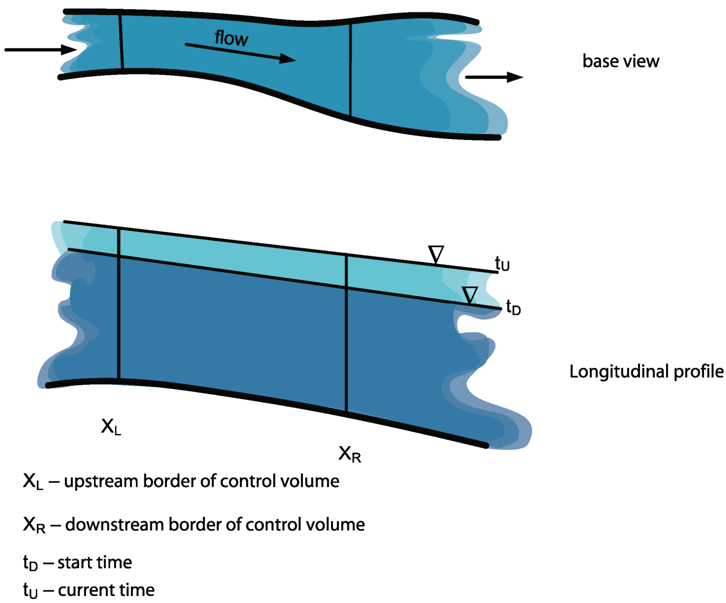


**Fig. 1.** Control volume

The integral form of the equation for the purpose of one-dimensional analysis is simple to define and the coefficients used to correct errors that occur due to the neglecting of the flow curvature are introduced in later stages.

The principle of mass conservation for the control volume may be expressed as follows:

$$\int_{X_L}^{X_R} [A(x,t_U) - A(x,t_D)]dx = \int_{t_D}^{t_U} [Q(x_L,t) + I(t) - Q(x_R,t)]dt \tag{1}$$

Time interval of integration is defined by times $t_U$ and $t_D$, so that $t_U > t_D$. The term $I(t)$ describes the increase in water volume due to lateral inflow. Water density is constant and for that reason can be eliminated from the equations; hence, the mass conservation principle may be also applied as the water volume conservation principle. Equation (1) defines change in water volume in the control volume as the difference between the volume that enters and the volume that leaves the control volume during the given time interval. The term $I(t)$ is called the lateral inflow and it is often encountered under natural conditions (inflow from the basin, inflow from the objects that return the water back to the system and similar).

**Principle of water momentum conservation.** In contrast to principle of mass and volume conservation, which account only for the discharge and volume of water, the principle of water momentum conservation also accounts for the flux of momentum and the forces that can appear on the boundaries of the control volume. The change in momentum is proportional to the force acting on the body and it is taking place in the direction of force action. In accordance with the momentum conservation principle, water momentum change inside the control volume, after certain period, must be equal to the downstream momentum in the same period increased by the momentum flux during the given period. The law on momentum conservation in the direction of x-axis can be presented in the following form:

$$\sum F_x = \frac{\partial}{\partial t}\int_{CV} \rho v_x \, \mathrm{d}\forall + \int_{CS} \rho v_x \mathbf{V}\cdot \mathrm{d}\mathbf{A} \tag{2}$$

where $F_x$ is the force acting on the control volume $CV$, $v_x$ is the velocity in the direction x, $\mathrm{d}\forall$ is the elementary volume, $\mathbf{V}$ is the velocity vector, and $\mathrm{d}\mathbf{A}$ is the elementary area represented as vector perpendicular to control area $CS$, which is bounding the control volume. The first term on the right-hand side of the equation represents the velocity of momentum change inside the control volume, while the second term is the flux of momentum.

In case of free-surface flow, the forces accounted for are the pressure forces that act on the upstream and downstream sides of the control volume, components of pressure on the riverbed bottom, as well as the gravity and friction forces. By shifting the term that describes the momentum inside the control volume to the left-hand side of the equation, of the term that represents the sums of forces to the right-hand side of the equation and the decomposition of the sum of forces, the law on momentum conservation in the control volume will have the following form:

$$\begin{aligned}\rho\int_{X_L}^{X_R}[Q(x,t_U)-Q(x,t_D)]dx = {} & \rho\int_{t_D}^{t_U}[\beta QV(x_L,t)-\beta QV(x_R,t)]dt \\ & +\rho g\int_{t_D}^{t_U}[J(x_L,t)+\int_{X_L}^{X_R}J_x^y dx - J(x_R,t)]dt \\ & +\rho g\int_{t_D}^{t_U}\int_{X_L}^{X_R}S_0 A dt dx \\ & -\int_{t_D}^{t_U}\int_{X_L}^{X_R}\tau P dt dx\end{aligned} \tag{3}$$

where $S_0$ is the slope of riverbed bottom and $\tau$ is the mean shear force stress of the flow against the contour of the river bottom.

Equation (3) represents a precise mathematical expression of the law on momentum conservation, where the momentum change is described through the momentum increase and downstream momentum change due to activity of all forces that act on the water in the control volume. Due to the adequate selection of forces and the introduced assumptions, Equation (3) is applicable to a control volume of arbitrary length.

Should the friction be generalized by the use of “friction drop” $S_f$ instead of bottom slope $S_0$, Equation (3) , upon dividing with $\rho$ , can be presented as follows:

$$\int_{X_L}^{X_R}[Q(x,t_U)-Q(x,t_D)]dx=\int_{t_D}^{t_U}[\beta QV(x_L,t)-\beta QV(x_R,t)]dt$$
$$+g\int_{t_D}^{t_U}[J(x_L,t)+\int_{X_L}^{X_R}J_x^y dx-J(x_R,t)]dt \tag{4}$$
$$+g\int_{t_D}^{t_U}\int_{X_L}^{X_R}A(S_0-S_f)dtdx$$

and it represents the integral form of momentum conservation equation for open-channel flows. This equation and the equations related to it are called “momentum equations”. The integral form of Equations (2) and (3) or (4) represents the foundation of all other forms of basic equations for unsteady open channel flows. It is worth mentioning that the friction force is the function of Manning's coefficient *n*, values of which are the subject of analysis in one of the following sections in the paper.

### *2.2. Equations of mass and momentum conservation*

Replacement of the integrals in Equations (2) and (3) by partial derivatives and the introduction of boundary conditions results in the following:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=q \tag{5}$$

and

$$\frac{\partial Q}{\partial t}+g\frac{\partial J}{\partial x}+\frac{\partial}{\partial x}(QV)=gA(S_0-S_f)+gJ_x^y \tag{6}$$

where $q$ represents the lateral inflow per unit length of the flow, which is defined as the function of the river station and time in the form of:

$$I(t)=\int_{x_L}^{x_R}q(x,t)dx \tag{7}$$

Equation (5) represents the law on mass conservation (where $\rho$ is constant) per flow length unit, and Equation (6) represents the law on momentum conservation per unit length of the flow. In the derivative of the discharge over time, the measure of the flow is momentum per unit length. The term on the right-hand side of the equation, comprised of the derivates of the value J, represents the net downstream pressure force per unit length. The derivative of QV value, which is shifted to the right-hand side of the equation, represents the net momentum drop per unit length. Finally, the value $gA(S_0-S_f)$ is the net downstream force per unit length, including the gravity and friction forces. Therefore, the Equation (6) defines the temporal changes of the momentum per unit length as a sum of net downstream forces and net momentum decrease.

### *2.3. Saint-Venant's equations*

By development of derivatives in the equations of mass and momentum conservation and its simplification the following is obtained:

$$A\frac{\partial V}{\partial t} + VT\frac{\partial y}{\partial x} + T\frac{\partial t}{\partial t}VA_x^y = q \tag{8}$$

and

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} = g(S_0 - S_f) - \frac{Vq}{A} \tag{9}$$

what represents the Saint-Venant's form of equations of flow (Saint-Venant 1871). This form and the others similar to it, represent the most common forms of equations of flow in hydraulics. The relation between the law on mass conservation and the law on momentum conservation is described by the Equations (8) and (9).

## 3. Boundary conditions in the model of unsteady flow

The first step in the hydraulic calculation is the decomposition of the hydro-system into a respective number of linear flows and control objects along the flow.

After the decomposition of the hydro-system, linear flows are linked in order to form the schematic model of the real system. Two linear flows may be linked by the junction object and the dam object, where the latter object implements the models of operation of a power plant and a spillway, as well as by a relevant number of control nodes, used for definition of boundary conditions. Direct linking of several flows creates a network of open-channel flows. An illustration of formation of a complex river system with different boundary conditions is presented in Section 4.2.

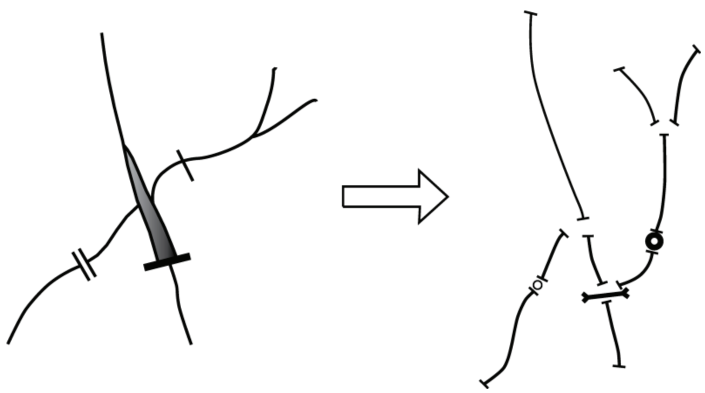

**Fig. 2.** System decomposition

River section represents an uninterrupted open-channel flow between two control objects. The flow within the river section is modeled by the basic Equations (1) and (4). River section is divided into the finite number of computational elements, whose ends are called nodes. To each node is assigned a relevant cross-section. Start and end nodes are called terminal nodes, while the other nodes are called internal nodes. Each river section has one upstream and one downstream terminal node, what determines the flow orientation. Each node in the system is related to the corresponding river station. Also, the elevation of the lowest point in the cross-section represents an important characteristic of that cross-section, because the couple of value stationary-elevation of the lowest point defines the riverbed bottom. The length of the computational element represents the absolute value of the difference between the stationary values of the two nodes that define it.

The water inflow into river section is defined as inflow at upstream cross-section and lateral inflow from catchment. Should the effect of inflow from the catchment area not be the subject of consideration, further defining of model characteristics related to lateral inflow is not required.

Control objects are of essential importance for the process of formation of the system of equation. The most important control objects are the junction object and the dam object.

Junction object represents the place where two or more river sections are joined to form a new section, which is the description of a confluence. Also, the junction can be used for modeling of branching of river flow (of the separation of river sections), i.e. modeling of the flow through river branches or the flow around river islands and similar.

The places with known level-grams or hydrographs represent a logical choice for the control objects, i.e. places where the boundary conditions are assigned. The location with a known discharge curve can also be a control object. Discharge curves cannot be assigned to starting nodes of the system, but they are used as the boundary condition in the downstream terminal node.

A control object of dam-type represents the sum of effects of objects located on the dam on the unsteady flow in the system. Calculation of operation of the power plant and the spillway is performed within the model that implements the dam object, what results in sum discharge through the dam objects. This discharge occurs as the internal boundary condition and will be described below.

### *3.1. Internal boundary conditions*

Flows (each is described by two water flow equations) are interrelated by special structures. These structures form the internal boundary conditions for solution of a system of equations which describe the flows and the links in the system. Each internal boundary condition links two or more flows because the special structure is composed of the junction between at least two flows, direct link and/or storage.

Internal boundary conditions are always related to terminal nodes of the flows, direct links and storages. Terminal nodes of the flows are marked as upstream or downstream. Node marks remain unchanged for any direction of water flow. The same rule applies to the internal boundary conditions; in some cases, this type of marking of the upstream and downstream node is suitable for easier referencing of nodes and utilization of values of their parameters. When an internal boundary condition is referred to, node mark is related to the special system structure that this particular condition is needed for. For example, a junction of two flows that consists of the two terminal nodes of these nodes is considered. This junction will be used for a simulation of a spillway dam. Description of the spilling over the dam marks one node in the junction as the upstream node and the other as the downstream one. In principle, marking of the node is not important; however, this mark has to be used consistently. Usually, the upstream node of the dam is also the downstream node of the flow that carries the water to the dam. The same applies to the downstream node of the dam which is the upstream node of the flow that carries the water away from the dam. In certain cases, in order to simplify the equations, this order has to be preserved, while for some special structures this order is not important.

Internal boundary conditions may be classified in two categories: those related to the mass conservation law and those related to water levels and discharges. The reason for this division is that mass conservation must be met in all junctions. In contrast to this, there are many options for the selection of a relation between the water level and the discharge.

**Mass (Volume) Conservation.** As mentioned above, dimensions of special structures described by internal boundary conditions are small relative to the flow dimensions; hence, change in water volume can be neglected for simulation purposes. According to the mass conservation law, the sum of discharges for each internal boundary condition must be equal to zero if discharge marks are correctly defined. Each terminal node of the flow must be assigned with the mark. The downstream node is positive and the upstream node is negative. Continuity equation (mass conservation equation) is then as follows:

$$\sum_{i=1}^{n_j} sign_i Q_{EX_i} = 0 \tag{10}$$

where $Q_{EX_i}$ is the discharge on i$^{th}$ terminal node of the flow, $n_j$ is the number of terminal nodes in the flow in the junction object and $sign_i$ - is the signum function for the terminal node of the flow ( $sign_i$ is -1 for the upstream, and +1 for the downstream end). Equation (10) is applied on each internal boundary condition.

**Equality of water levels.** In the simplest relation that relates two system nodes, water levels at the flow ends must be equal as any time and for all discharges:

$$z_{w_L} - z_{w_R} = 0 \tag{11}$$

where indices $L$ and $R$ denote two terminal nodes of the flow. This relation is useful for a simple junction object.

**Power plants and spillways.** Application of internal boundary condition provides for elegant modeling of characteristics of power plants and spillways on dams. The discharge through the power plant and over the spillway can be the function of headwater and tailwater water levels, as well as other parameters that define the mode of their operation.

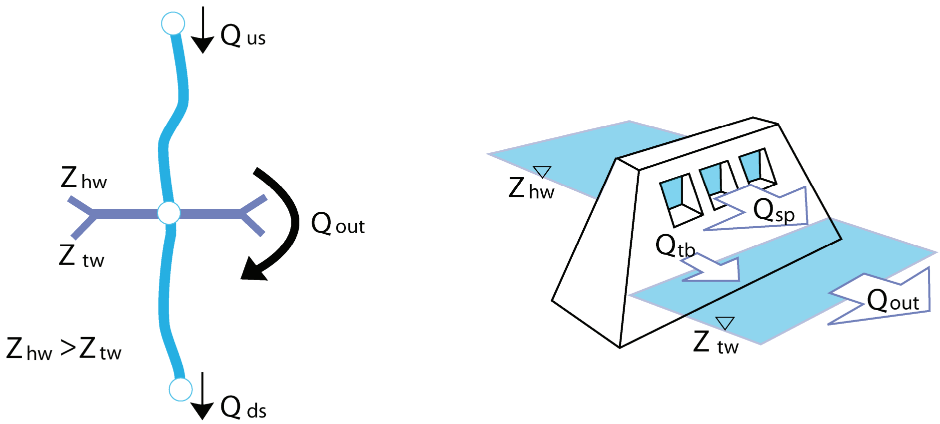


**Fig. 3.** Internal boundary conditions: dam with power plant and spillway

If the power plant or the spillway are represented by the control structures in the model, than the headwater level is equal to the level in the upstream boundary node, while the tailwater level is equal to the level in the downstream boundary node. The upstream boundary node of the control structure (power plant or spillway) is also the downstream boundary node of the flow which carries the water into it, while the downstream boundary node of the control structure is also the upstream boundary node of the flow which carries the water away from that control structure. The values of headwater and tailwater levels, as well as of other parameters that affect the operation of the power plant and the spillway, are read at each time step. The discharge is determined upon the obtained values and the function describing the operation of the power plant or the spillway. The discharge obtained in this manner is assigned to one or both boundary nodes of the control structure. If discharge is assigned only to one boundary node, than an additional internal boundary condition must be applied, which should ensure mass conservation.

This method of modeling of the power plant and the spillway is particularly suitable, because it provides for defining of the arbitrary function of behavior of these objects. This also includes the possibility to define non-analytical forms, such as different decision-making methods, and inclusion of different parameters that could impact the operation of the power plant and the spillway.

### *3.2. External boundary conditions*

External boundary conditions must be assigned to all terminal nodes which are not linked with the special structures. There is a possibility of assigning three types of external boundary conditions: discharge as the function of water level (by the means of discharge curves), discharge as the function of time (by them means of hydrographs) and water level as the function of time (by the means of level-grams).

**Discharge as a function of water level.** Each control structure that consists of one node may serve as an external boundary condition. These structures represent the discharge as the function of water level. The only difference is that with the external boundary conditions, the node in which water level is read and the node where the discharge s defined must be the same. In the case of internal boundary conditions, they can be different or the same, depending on the given situation. By taking this difference into account, internal boundary conditions may be applied as external.

Discharge as the function of the water level cannot be applied as the upstream boundary condition, because the discharge would in that case increase without any limitation during the calculation. Due to discharge increase in the upstream boundary condition that result would cause level increase, and this increase in discharge would also cause an increase in the discharge from the one-node control structure. This is the reason why this function may be used as the downstream boundary condition.

**Discharge as the function of time.** Discharge as the function of time in an external node is marked as $f_{qb}(t)$, where $qb$ denotes the external node. The equation for discharge as the function of time is as follows:

$$Q_{qb} - D_D f_{qb}(t) = 0 \tag{12}$$

where $D_D = 1$, if positive values $f_{qb}(t)$ represent inflow of water into the system in the external boundary node, and $D_D = -1$, if positive values $f_{qb}(t)$ represent the water that flows out of the system in this node. This method of discharge marking can be found in all boundary conditions of the model. Inappropriate use of this type of boundary conditions may result in severe errors.

Discharge as the function of time is usually assigned to the upstream terminal node of the flow as the external boundary condition. As a consequence, the downstream conditions exert no impact on the discharge in the given node. If the impact of the downstream part is apparent, the boundary condition will have to be moved upstream to the region outside the impact of the downstream part. Secondly, if the discharge is assigned as the function of time both on the upstream and the downstream node, all differences between these discharges will in the calculation have impact on the water level.

**Water level as the function of time.** If the value of discharge in the external node corresponds to the steady hydraulic regime, than the equation of water level as the function of time is as follows:

$$z_{w_b} - f_z(t) = 0 \tag{13}$$

where $z_{w_b}$ is the water level in the external boundary node and $f_z(t)$ is the imposed water level in the external boundary node.

**Initial conditions.** Before the start of the unsteady-flow simulation, the initial values of discharge and water must be known for each node in the model. These values are calculated by the steady-flow calculation. This calculation uses steady-flow equations derived by simplification of the unsteady-flow equations. The majority of control structures are not covered by this calculation because the characteristics of control structures mainly relate to unsteady flow. After the steady-flow calculation has defined the initial condition, the unsteady-flow calculation can start.

## 4. Procedure of numerical solving

### *4.1. Finite differences method*

Four points in the x-t graph are used for the definition of the space of the (approximate) integration of Equations (1) and (4), in order to create a system of algebraic equations for the subject section. These four points are shown in Figure 4. The equations are written for the control volume that corresponds to the calculation element in the section between the adjacent cross-sections.

The method of approximate integration used here is the weighting trapezoid method, which can be described as follows:

$$\int_a^b f(x)dx \approx (b-a)\left[(1-W)f(a) + W f(b)\right] \tag{14}$$

where $a$ and $b$ are the integration boundaries and $W$ is the weighting coefficient of the function at the upper boundary of the integral. The coefficients can have values $0 \le W \le 1$.

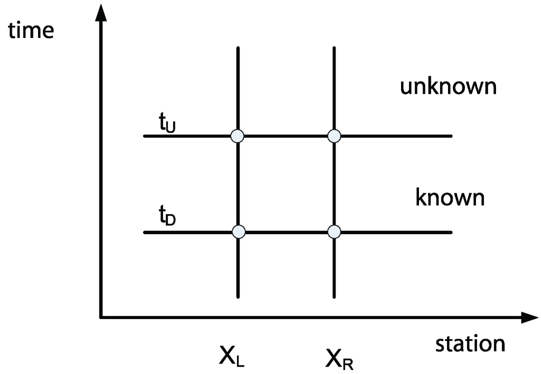


**Fig. 4.** Four points in the x-t plane

*4.1.1. Mass conservation*

Let $\Delta x$ be the length of the calculation element and $\Delta t$ the time increment. If approximate integration of Equation (14) is performed, the result is as follows:

$$\begin{aligned}&\Delta x\left[\left(1-W_{A_U}\right)M_{A_{LU}}A_{LU}+W_{A_U}M_{A_{RU}}A_{RU}\right]-\Delta x\left[\left(1-W_{A_D}\right)M_{A_{LD}}A_{LD}+W_{A_D}M_{A_{RD}}A_{RD}\right]\\&-\Delta t\left[Q_{LD}+W_T\left(Q_{LU}-Q_{LD}\right)\right]+\Delta t\left[Q_{RD}+W_T\left(Q_{RU}-Q_{RD}\right)\right]-\Delta t I_M=0\end{aligned} \tag{15}$$

where index $M$ denotes the mean value and all terms are transferred to the left-hand side of the equation before approximation.

The mean area at time $t_U$ is determined by the weighting coefficient $W_{A_U}$ and the mean area at time $t_D$ by the other coefficient, $W_A$, the value of which changes over time. The algebraic form of the mass conservation equation for a given rectangle in the x-t plane is represented by Equation (15).

*4.1.2. Momentum conservation*

The momentum conservation equation, Equation (4), may be simplified if the integral that represents friction on the boundary area and weight and pressure forces is approximated in relation to length:

$$\int_{x_L}^{x_R} A\left[S_f+\frac{(\partial z_w)}{\partial x}\right]dx \approx A_M\left(\Delta x S_{f_M}+\Delta z_w\right) \tag{16}$$

where index $M$ denotes the mean value. All mean values in Equation (16) are the functions of space coordinate – river station, and the time is fixed.

A further simplification of this expression is performed if the approximation of the integrals of the terms related to pressure, weight and friction is represented by $P_{GF}$, so that the following applies:

$$P_{GF}=A_M\left(\Delta x S_{f_M}+\Delta z_w\right) \tag{17}$$

The term $A_M$ that represents the mean area, is obtained as the interpolated value between the values of stationary $x_L$ and $x_R$, with the weighting coefficient that corresponds to the linear interpolation of geometry $W_A$:

$$A_M = A_L + W_A \left( A_R - A_L \right) \tag{18}$$

and the mean “friction drop” is:

$$S_{f_M} = \frac{(Q_M |Q_M|)}{K_M} \tag{19}$$

where the mean discharge is $Q_M = Q_L + \frac{1}{2}(Q_R - Q_L)$, and the interpolated (mean) discharge module $K_M = K_L + W_x(K_R - K_L)$. In present notation, the integral of pressure, gravitation and friction terms per unit length of the computational element at time $t_U$ is denoted with $P_{GF_U}$ that is $P_{GF_D}$ for time $t_D$ - the main index is supplemented by the time mark that represents the integration time.

The application of arithmetic averaging to the momentum in the control volume and the application of approximation in relation to time result in the Equation similar to Equation (4), with the weighting coefficient for time dimension $W_T$, in the following form:

$$\begin{aligned}
&\frac{\Delta x}{2}\left[ M_{Q_{LU}} Q_{LU} + M_{Q_{RU}} Q_{RU} \right] - \frac{\Delta x}{2}\left[ M_{Q_{LD}} Q_{LD} + M_{Q_{RD}} Q_{RD} \right] \\
&-\Delta t \left[ \beta_{LD} Q_{LD} V_{LD} + W_T \left( \beta_{LD} Q_{LU} V_{LU} - \beta_{LD} Q_{LD} V_{LD} \right) \right] \\
&+\Delta t \left[ \beta_{RD} Q_{RD} V_{RD} + W_T \left( \beta_{RU} Q_{RU} V_{RU} - \beta_{RD} Q_{RD} V_{RD} \right) \right] \\
&+g\Delta t \left[ P_{GF_D} + W_T \left( P_{GF_U} - P_{GF_D} \right) \right] + \Delta t F_{DEC_M} = 0
\end{aligned} \tag{20}$$

where $F_{DEC_M}$ is the equivalent deceleration due to obstacles in the flow and losses in vortices due to non-prismatic shape of the riverbed, averaged in relation to time.

### *4.2. Matrix forms of model components*

Previously defined equations describe the discharges and water levels in the network of open channels. The nature of the river system and user's choices determine which equations will be included in the river system model. All selected equations must be solved simultaneously; hence they form a system of non-linear equations that should be solved by some of standard methods. In a general case, the application of some of the numeric methods for solving of the nonlinear equations, such as the Newton iteration method, is necessary. The application of that method reduces the solving of the system of non-linear equations to the solving of the system of linear equations through the series of iterations.

Several simple examples are provided below to illustrate the matrix form of the system of equation that describes the network of open channels. These examples are shown starting with the simple ones, in order to present the development of the Jacobian matrix for complex system.

#### *4.2.1. Matrix forms of the river section*

The simplest example includes one river section with the upstream boundary condition assigned in the form of discharge as the function of time (hydrograph) and the downstream boundary condition assigned in the form of discharge as the function of level (discharge curves). Figure 5 shows this river section with 4 nodes in total, two terminal and two internal ones.

The system of non-linear equations in this example consists of 8 equations and 8 unknowns, two for each node. The unknown values are the values of level and discharge in all nodes. The order of equations in the matrix is very important, because it affects the structure of the resulting Jacobian matrix. The matrix is calculated in each step and the system of linear equations is solved. If the Jacobian matrix is well-ordered, the solution is efficient.

The order of equations order in this example is as follows: the equation for the upstream boundary node is written first, followed by six equations for three calculation elements inside the section or, to be precise, by three pairs of equations of mass and momentum conservation. The last one is the equation for the downstream boundary node. This method of writing of equations results in the band Jacobian matrix. The equation for the upstream boundary condition includes only two unknowns, $Q_1$ and $Z_1$, wherein the direction of node numbering is towards the downstream end. The Jacobian matrix includes partial derivatives, $F'$, residual functions of each momentum equation and each equation for the external boundary conditions.

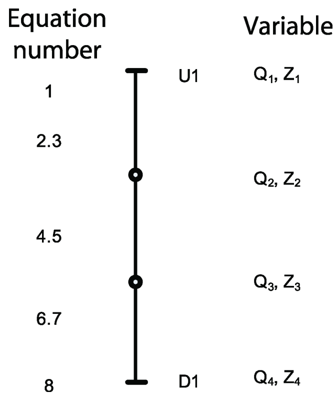


**Fig. 5.** Simple riverbed with one river section

Consequently, the remaining seven terms in the first row of the Jacobian matrix are equal to zero, because those seven variables do not appear in the first equation. Only four unknowns are included in each pair of equations for computational elements, for discharge and water level in each node. Therefore not more than four terms in each row are not equal to zero. Finally, the last equation (for the downstream boundary condition) includes only discharge and water level in the downstream terminal node. This is the reason why not more than two terms in each row have values other than zero. A common linear system of equations written in matrix form is of the following form:

$$\begin{bmatrix} F'_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{21} & F'_{22} & F'_{23} & F'_{24} & 0 & 0 & 0 & 0 \\ F'_{31} & F'_{32} & F'_{33} & F'_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & F'_{43} & F'_{44} & F'_{45} & F'_{46} & 0 & 0 \\ 0 & 0 & F'_{53} & F'_{54} & F'_{55} & F'_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & F'_{65} & F'_{66} & F'_{67} & F'_{68} \\ 0 & 0 & 0 & 0 & F'_{75} & F'_{76} & F'_{77} & F'_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & F'_{87} & F'_{88} \end{bmatrix} \begin{bmatrix} \Delta Q1 \\ \Delta Z1 \\ \Delta Q2 \\ \Delta Z2 \\ \Delta Q3 \\ \Delta Z3 \\ \Delta Q4 \\ \Delta Z4 \end{bmatrix} = \begin{bmatrix} -F_1 \\ -F_2 \\ -F_3 \\ -F_4 \\ -F_5 \\ -F_6 \\ -F_7 \\ -F_8 \end{bmatrix}.$$

**Fig. 6.** Matrix Form of equations that describe the flow along one river section

The method of index writing is shown in Figure 6. Indices for partial derivatives are formed in such a manner that the first number marks the residual function, and the other one the variable value. The rule applied here is as follows: an odd index marks the discharge, while an even index marks the water level. In equations for the river section, the mass conservation equation always comes before the momentum conservation equation.

Consequently, besides the equations describing the boundary conditions, all other equations line up in such a manner that the mass conservation equation should be marked with an even number, while the momentum conservation equation should be marked with an odd number. For example, $F'_{53}$ is the partial derivative of the residual function of the momentum conservation equation for the second element per third variable, i.e. per discharge in the second node. Figures 5 and 6 show that index 5 refers to the second equation (momentum conservation) of the second element, and index 3 to the discharge in the node 2.

The width of the band in this example is 5. All sections in which the basic algebraic equations correspond to the previous ones will have this band width in the Jacobian matrix. In this example out of the total 64 Jacobian elements, only 27 are other than zero. However, if, for example, a section has 50 nodes, than the Jacobian matrix will have 10.000 elements, out of which 395 are other than zero.

The number of equations is denoted with $ne$ and band width with $m$. The required number of computational operations for solving of the linear system of an arbitrary structure is proportional to $ne^3$. If the system of equations is of the band-type that the required number of computational operations is proportional to $nem^2$. In this example $m$ is equal to 5. In a complex example with 50 nodes per section and 100 equations, the number of computational operations for the band structure amounts to only 0.25% of the number of operations for an arbitrary structure.

### *4.2.2. Matrix form of network of flows*

In case of more than two river sections (confluence or flow branching), the band width is not constant. The simplest example of this is the confluence of two sections, as shown in Figure 7. The confluence of two sections is a simple junction, where the water levels are equal in all three nodes.

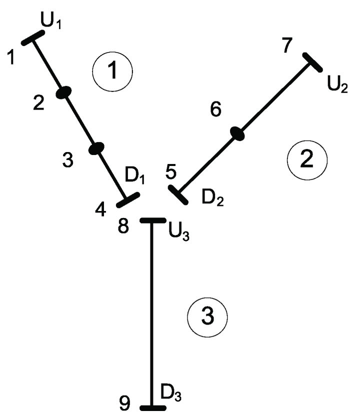


**Fig. 7.** Network of flow that consists of three river sections

The Jacobian matrix for the model of the confluence shown in Figure 7 has the following form:

Columns

| Rows | $Q_1$ | $Z_1$ | $Q_2$ | $Z_2$ | $Q_3$ | $Z_3$ | $Q_4$ | $Z_4$ | $Q_5$ | $Z_5$ | $Q_6$ | $Z_6$ | $Q_7$ | $Z_7$ | $Q_8$ | $Z_8$ | $Q_9$ | $Z_9$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | | | | $Q_1=Q_1(t)$ | | $Q_1(t)$ |
| 2 | X | X | X | X | | | | | | | | | | | | | | | Continuity 1-2 | | |
| 3 | X | X | X | X | | | | | | | | | | | | | | | Pres.mom. 1-2 | | |
| 4 | | | X | X | X | X | | | | | | | | | | | | | Continuity 2-3 | | |
| 5 | | | X | X | X | X | | | | | | | | | | | | | Pres.mom. 2-3 | | |
| 6 | | | | | X | X | X | X | | | | | | | | | | | Continuity 3-4 | | |
| 7 | | | | | X | X | X | X | | | | | | | | | | | Pres.mom. 3-4 | | |
| 8 | | | | | | | | 1 | | -1 | | | | | | | | | $Z_4=Z_5$ | = | 0 |
| 9 | | | | | | | 1 | 0 | 1 | 0 | | | | | -1 | | | | $Q_4+Q_5=Q_8$ | | 0 |
| 10 | | | | | | | | | X | X | X | X | | | 0 | | | | Continuity 6-5 | | |
| 11 | | | | | | | | | X | X | X | X | | | 0 | | | | Pres.mom. 6-5 | | |
| 12 | | | | | | | | | | | X | X | X | X | 0 | | | | Continuity 7-6 | | |
| 13 | | | | | | | | | | | X | X | X | X | 0 | | | | Pres.mom. 7-6 | | |
| 14 | | | | | | | | | | | | | 1 | 0 | 0 | | | | $Q_7=Q_7(t)$ | | $Q_7(t)$ |
| 15 | | | | | | | | | | 1 | 0 | 0 | 0 | 0 | 0 | -1 | | | $Z_5=Z_8$ | | 0 |
| 16 | | | | | | | | | | | | | | | X | X | X | X | Continuity 8-9 | | |
| 17 | | | | | | | | | | | | | | | X | X | X | X | Pres.mom. 8-9 | | |
| 18 | | | | | | | | | | | | | | | | | X | X | $Q_9=Q_9(Z_9)$ | | |

**Fig. 8.** Matrix form

Indices for $Q$ and $Z$ correspond to the locations shown in the above figure. Mark "X" denotes the terms with a value unequal to zero and terms equal to zero outside the band are omitted. Column titles correspond to the variables that appear in them. In rows 1 and 14 it can be seen how discharges were assigned as the external boundary conditions. The equality of water levels in all three branches of the branching is achieved by the means of equations in rows 8 and 15, while the mass conservation equation of the branching is given in equation 9. In order to make the system of equations definite, it is necessary to introduce the equation in the row 18, as a dependency of discharge upon water level (discharge curve) in node 9.

Figure 8 shows that this form resembles the band form. Now the equation in row 15 and the column marked with $Q_8$ is analyzed. The band shape is disturbed here and the terms unequal to zero are rather far from the main diagonal. If the section (2) also contained more computational elements, the structure would be even more complex. This matrix is called the profile matrix and solving of this matrix is somewhat more demanding that the solving of the band matrix.

*4.2.3 Matrix form of introduction of the control object*

Present example introduces the object of a low spillway dam between two sections. The schematic representation of this system is shown in Figure 9. Section (1) has two and section (2) three nodes. Therefore, the total number of nodes is 5 and the number of unknown values is 10.

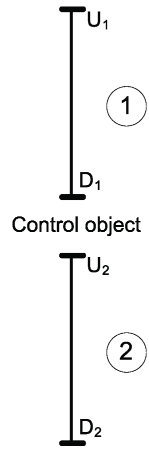


**Fig. 9.** Simple network with the control object

The Jacobian matrix for the case presented in Figure 9 has the form:

Columns

| Rows | $Q_1$ | $Z_1$ | $Q_2$ | $Z_2$ | $Q_3$ | $Z_3$ | $Q_4$ | $Z_4$ | $Q_5$ | $Z_5$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | $Q_1=Q_1(t)$ | | $Q_1(t)$ |
| 2 | X | X | X | X | | | | | | | Continuity 1-2 | | |
| 3 | X | X | X | X | | | | | | | Pres.mom. 1-2 | | |
| 4 | | | X | X | | X | | | | | $Q_2=Q_2(Z_2, Z_3)$ | | |
| 5 | | | 1 | 0 | -1 | 0 | | | | | $Q_2=Q_3$ | = | 0 |
| 6 | | | | | X | X | X | X | | | Continuity 3-4 | | |
| 7 | | | | | X | X | X | X | | | Pres.mom. 3-4 | | |
| 8 | | | | | | | X | X | X | X | Continuity 4-5 | | |
| 9 | | | | | | | X | X | X | X | Pres.mom. 4-5 | | |
| 10 | | | | | | | | | X | X | $Q_5=Q_5(Z_5)$ | | |

**Fig. 10.** Matrix form of equations for the system with the control object

The upstream and downstream boundary conditions are the same as the ones in Section 4.2.2.; hence, the forms of the first and the last equation are identical. Internal boundary conditions on the dam are the mass conservation equation at the junction and the two-node control object for the discharge over the dam. Only two terminal nodes are included here. Consequently, only 2 out of 10 terms in the row of the Jacobian that is related to the internal boundary condition will be unequal to zero. The two-node control object may include the discharge on one of the terminal nodes or the water level on both terminal nodes. Therefore, this row of the Jacobian contains not more than three terms whose value of which is unequal to zero. The remaining rows of the Jacobian are formed by the means of the methods described above.

The row 4 of this matrix shows the method of assigning of the discharge dependence on headwater and tailwater water levels. The equation of mass (volume) conservation on the dam is given in row 5. It shows that internal boundary conditions do not change the band width, meaning that they exert no impact on calculation performance.

## 5. Matrix form of the system “Iron Gate 1” HPP – “Iron Gate 2” HPP

The model of the system “ Iron Gate 1” HPP – “ Iron Gate 2” HPP, as presented in Divac et al. (2009), is composed of the series of river sections that represent the River Danube course from the city of Novi Sad to the Timok River confluence, including all its major tributaries (Figure 11).

Special attention was paid to modeling of the system of hydropower plants “Iron Gate 1” – “Iron Gate 2” – “Gogoš”, whose effect on the flow is represented by the internal boundary conditions.

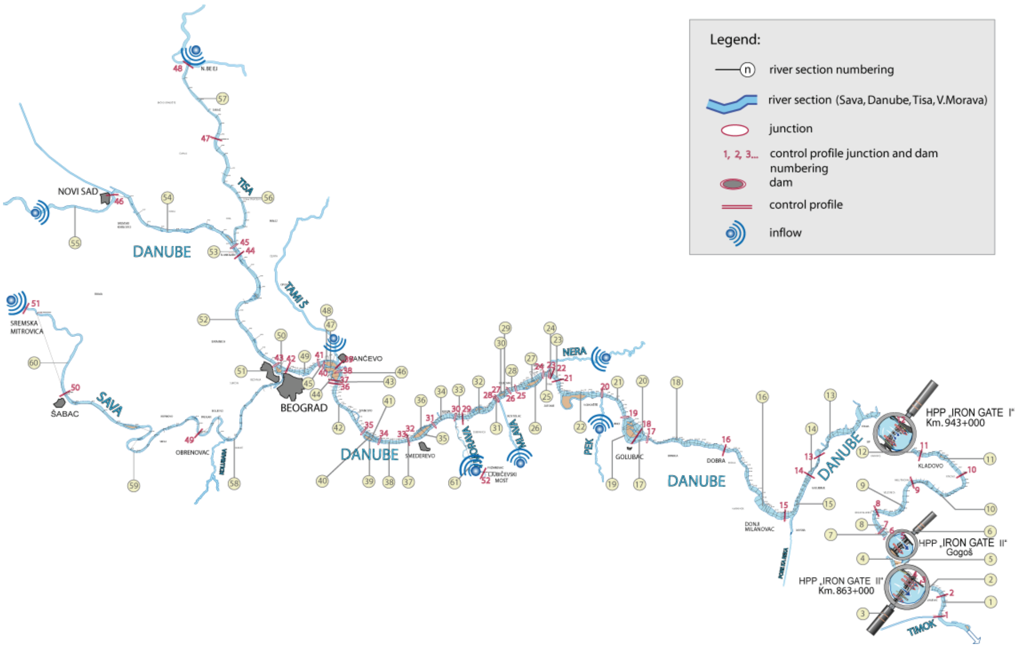


**Fig. 11.** River Danube course with the system “ Iron Gate 1” HPP - “ Iron Gate 2” HPP – ”Gogoš” HPP

Figure 12 shows that 9 kilometers upstream from the city Kladovo Danube River is intersected by the “Iron Gate 1” hydropower plant, and that 56 kilometers downstream from Kladovo the river branches into two flows. The “Gogoš” power plant is located at the very start of the smaller of the two flows and the “Iron Gate 2” hydropower plant is located at the thirteenth kilometer along the larger of the two flows. Two kilometers downstream from the “Iron Gate 2” hydropower plant the two flows are rejoined into a single flow. This system of dams and hydropower plants has a major impact on the flow characteristics in river sections; hence, the modeling of that part of river section be a subject of due attention.

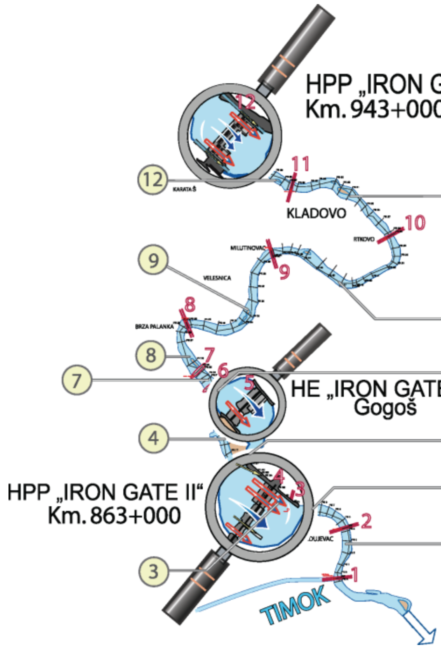


**Fig. 12.** Enlarged subsystem of "Iron Gate" hydropower plants

Discretization in the real model requires a higher degree of detail and, accordingly, a larger number of sections, for the model to provide realistic simulation in dynamic regimes. However, in order to present the essence of the model in the most interesting zone (the zone with dams), it is suitable to disregard the importance of discretization degree and to try to use the minimum number of sections, as to show the matrix link that results from the mathematical approach applied to the model of the complete system. A simplified model of the system is an illustration of the use of the minimum number of sections with the minimum number of nodes (two per section) for modeling of river flow. Section (1) is located upstream from the first control object ("Iron Gate 1" dam), section (2) runs downstream up to the branching, section (3) represents the main flow up to the second control object ("Iron Gate 2" dam), section (4) runs downstream under the second control object, section (5) runs to the branching under the first control object up to the third control object ("Gogoš" dam), section (6) routes the water from the third control object up to the confluence downstream from the second control object and the last section (7) runs from the node where the downstream flows of the second and third control objects are joined to the system exit (Figure 13). The total number of nodes is 14 and the number of unknown values in the nodes is 28.

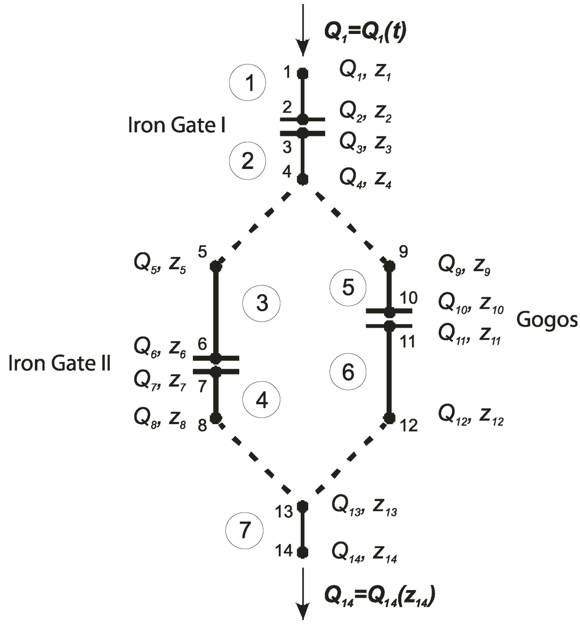


**Fig. 13.** System of control objects

The upstream boundary condition is the known series of $Q_1 = Q_1(t)$, and the downstream boundary condition is the discharge curve $Q_{14} = Q_{14}(Z_{14})$; this dictates the form of the first and the last equation of the system.

Two equations (the continuity equation and the momentum conservation equation) are generated for each section (between each two nodes), which is denoted by "X" (value other than zero) in the system matrix shown in Figure 14.

The broken lines denote the model parts that represent the overlapped nodes for which mathematical links are defined that correspond to the boundary conditions in case of a branching or a confluence. This relates to the equality of water levels in overlapped nodes, as well as to the balance of discharges that entering into and exit from the overlapping nodes.

Internal boundary conditions on the dams are the equation of mass conservation at junctions and the two-node control object for the discharge over the dam for each control object in the model. Only two terminal nodes are included here; hence, only 2 of all terms in the row of the Jacobian of the system related to the internal boundary condition will be unequal to zero. The two-node control object may include the discharge on one of the terminal nodes or water levels on both terminal nodes. In this concrete case (with power plants and spillways), the discharge is the function of headwater and tailwater water levels (and management rules which are the internal system function), i.e. the water level in the nodes that represent the control object.

| | $Q_1$ | $Z_1$ | $Q_2$ | $Z_2$ | $Q_3$ | $Z_3$ | $Q_4$ | $Z_4$ | $Q_5$ | $Z_5$ | $Q_6$ | $Z_6$ | $Q_7$ | $Z_7$ | $Q_8$ | $Z_8$ | $Q_9$ | $Z_9$ | $Q_{10}$ | $Z_{10}$ | $Q_{11}$ | $Z_{11}$ | $Q_{12}$ | $Z_{12}$ | $Q_{13}$ | $Z_{13}$ | $Q_{14}$ | $Z_{14}$ | Equation |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | $Q_1=Q_1(t)$ |
| 2 | X | X | X | X | | | | | | | | | | | | | | | | | | | | | | | | | Continuity 1-2 |
| 3 | X | X | X | X | | | | | | | | | | | | | | | | | | | | | | | | | Pres.mom. 1-2 |
| 4 | | | 1 | 0 | 0 | 0 | | | | | | | | | | | | | | | | | | | | | | | $Q_2=Q_2(Z_2,Z_3)$ |
| 5 | | | 1 | 0 | -1 | 0 | | | | | | | | | | | | | | | | | | | | | | | $Q_2=Q_3$ |
| 6 | | | | | X | X | X | X | | | | | | | | | | | | | | | | | | | | | Continuity 3-4 |
| 7 | | | | | X | X | X | X | | | | | | | | | | | | | | | | | | | | | Pres.mom.3-4 |
| 8 | | | | | | | 1 | | -1 | | | | | | | | -1 | | | | | | | | | | | | $Q_4=Q_5+Q_9$ |
| 9 | | | | | | | | 1 | | -1 | | | | | | | | | | | | | | | | | | | $Z_4=Z_5$ |
| 10 | | | | | | | | 1 | | | | | | | | | | -1 | | | | | | | | | | | $Z_4=Z_9$ |
| 11 | | | | | | | | | X | X | X | X | | | | | | | | | | | | | | | | | Continuity 5-6 |
| 12 | | | | | | | | | X | X | X | X | | | | | | | | | | | | | | | | | Pres.mom.5-6 |
| 13 | | | | | | | | | | | 1 | 0 | 0 | 0 | | | | | | | | | | | | | | | $Q_6=Q_6(Z_6,Z_7)$ |
| 14 | | | | | | | | | | | 1 | 0 | -1 | 0 | | | | | | | | | | | | | | | $Q_6=Q_7$ |
| 15 | | | | | | | | | | | | | X | X | X | X | | | | | | | | | | | | | Continuity 7-8 |
| 16 | | | | | | | | | | | | | X | X | X | X | | | | | | | | | | | | | Pres.mom.7-8 |
| 17 | | | | | | | | | | | | | | | 1 | | | | | | | | 1 | | -1 | | | | $Q_8+Q_{12}=Q_{13}$ |
| 18 | | | | | | | | | | | | | | | | 1 | | | | | | | | | | -1 | | | $Z_8=Z_{13}$ |
| 19 | | | | | | | | | | | | | | | | | X | X | X | X | | | | | | | | | Continuity 9-10 |
| 20 | | | | | | | | | | | | | | | | | X | X | X | X | | | | | | | | | Pres.mom.9-10 |
| 21 | | | | | | | | | | | | | | | | | | | 1 | 0 | 0 | 0 | | | | | | | $Q_{10}=Q_{10}(Z_{10},Z_{11})$ |
| 22 | | | | | | | | | | | | | | | | | | | 1 | 0 | -1 | 0 | | | | | | | $Q_{10}=Q_{11}$ |
| 23 | | | | | | | | | | | | | | | | | | | | | X | X | X | X | | | | | Continuity 11-12 |
| 24 | | | | | | | | | | | | | | | | | | | | | X | X | X | X | | | | | Pres.mom.11-12 |
| 25 | | | | | | | | | | | | | | | | | | | | | | | | 1 | | -1 | | | $Z_{12}=Z_{13}$ |
| 26 | | | | | | | | | | | | | | | | | | | | | | | | | X | X | X | X | Continuity 13-14 |
| 27 | | | | | | | | | | | | | | | | | | | | | | | | | X | X | X | X | Pres.mom.13-14 |
| 28 | | | | | | | | | | | | | | | | | | | | | | | | | | | X | X | $Q_{14}=Q_{14}(Z_{14})$ |

**Fig. 14.** Matrix formulation of the system of control objects

## 6. Simulation algorithm

Upon reading of input data and their possible testing, the variables that shall remain throughout the simulation are initialized; the variables that are not constant throughout the simulation, but their values change from one step to another or during the iterative procedure of solving of system equations are initialized too.

The calculation of the first time step of the simulation is performed after these preparations of the calculation. Before the start of solving of all time steps it is necessary to prepare all parameters that are variable in time and which do not describe flow. In order to determine all system parameter for the corresponding time moment, it is necessary to assume the system state at the start of the time step.

In the case of the first simulation step, the initial state will be determined by the calculation of the steady flow (i.e. by the calculation of the water level line in the steady-flow regime), or based upon known values (determined by measurement or by previous simulation). This is a method to determine the values of discharge and water level on all transversal profiles of the computational section. The initial values for other steps are obtained by the calculations of previous steps.

The initial parameter values for the dam with the power plant and the spillway are determined in a similar manner. The discharge required on the power plant is determined upon the current values of headwater and tailwater water levels and the demanded power, by using the parameter curves. Also, the discharge on the spillway is determined upon the spillway characteristics.

After the characteristics of the whole model for a given time step have been determined, the next phase is the iterative solution of a system of non-linear equations by the Newton method. The solutions obtained in each iteration are compared to defined values; then, based on the difference between them, the terms in the Jacobian matrix are corrected. Since the operation of the active hydropower objects (dams with the power plant and spillway) is dependent upon the current values of headwater and tailwater water levels, the outflow from the subject objects must also be corrected. The values corrected in this manner are used for repeating of the whole procedure of solving of the system equations until the convergence criterion is satisfied. In addition to convergence, during the solution of the system of non-linear equations it is necessary to meet the demands related to electricity generation.

After the convergence has been reached, the obtained solutions are printed; these solutions include the water levels and discharges on all control profiles and then the next step of the solution starts, until the final simulation time is reached.

The simulation algorithm based on the calculation of unsteady 1D flow is shown in Figure 15.

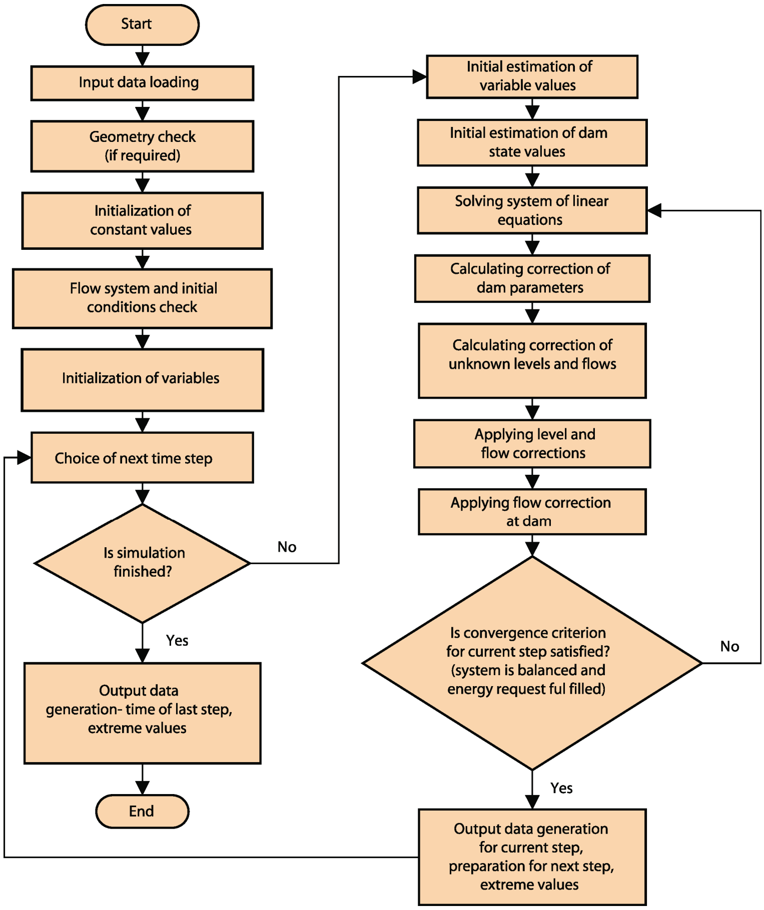


**Fig. 15.** Simulation algorithm

## 7. Model parameters and calibration

Modeling of the phenomenon of unsteady flow implies a complex interaction of the fluid and the environment. All models attempt to describe the processes by the finite number of mathematical expressions, the parameters of which have to be determined repeatedly for each concrete problem. Regardless of the model type, it is to a certain extent concentrated in certain point in space and certain point in time. Therefore, the majority of parameter values cannot be determined by measured, but they have to be evaluated by indirect methods.

In order to determine the model parameters that reflect the actual system in the best possible way, it is necessary to calibrate the model before it is used, by using one of the calibration methods. Calibration of parameters means the determination of values of parameters that allow for the results obtained by the model that deviate minimally from the measured values. Calibration requires the measured values of system inputs and the parameters used for system management (the inflows, the outflows on dams and similar), as well as the corresponding system outputs (usually the realized water levels on several profiles).

The only free parameter within the model, the value of which is to be evaluated, is the Manning's coefficient that depends upon discharge:

$$n_i = f(Q) \tag{21}$$

where:

$i$ - ordinal number of the space with a constant coefficient and

$Q$ - water discharge.

It was assumed that in the mathematical model of the “Iron Gate 1” system and the “Iron Gate 2” system it is necessary to define the curve $n(Q)$ for each river section on River Danube or its tributaries that has certain physical-geographical characteristics (according to the position of tributaries or according to flow characteristics).

The calibration is based on the determination of the values of the parameters of the simulation model for the “Iron Gate 1” system and the “Iron Gate 2” system, which lead to the minimization of the difference between the water levels on the control profiles resulting from the simulation and the measured water levels.

The first step in this process is the selection of several periods with the duration of 7 days (as the predefined longest simulation time) that shall be used for simulations and comparison of calculated water levels with the measured values. Then, the initial values of the Manning's coefficient have been defined upon the previous experience in system modeling.

The calibration of model parameters is performed simultaneously for all selected periods. For each of the selected periods the simulation is performed upon the single recommended set of Manning's coefficients. The choice of the most adequate set of coefficients is performed according to the adopted criterion, which is used for evaluation of the quality of each parameter set upon the deviation of simulated values from the respective measurements. The verification of the quality of model parameters must be performed using the period different from the one used for calibration.

Within the period selected for calibration, from January 1st, 2006 to March 26th, 2006, seven 7 day long periods were formed and model simulation was performed. Figure 16 shows that this period also covers major changes in inflow, as well as quasi-stationary regimes, with discharges between 3000 and 4500 $m^3/s$.

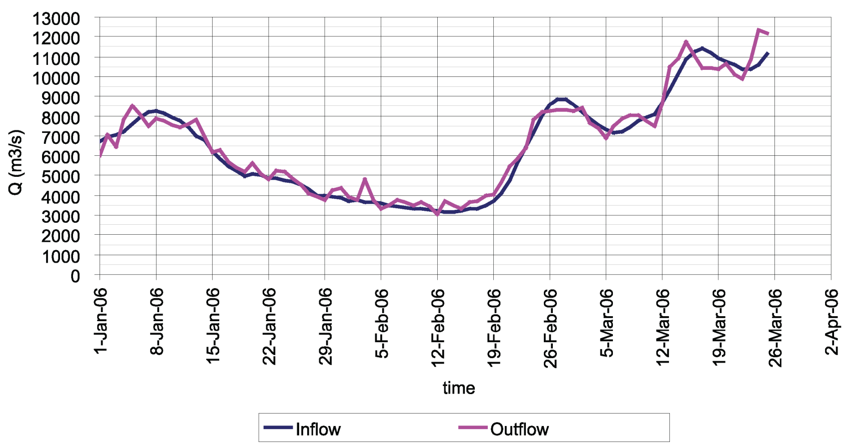


**Fig. 16.** Inflow to storage and outflow on the “Iron Gate 1” dam profile during the period selected for calibration

The curves $n = f(Q)$, which were determined by the steady-flow model calibration for the same morphologic state during the preparation of the document “Calculation of regulated water levels in the “Iron Gate 1” HPP storage, from the year 2002, were adopted as the approximate values of the Manning's coefficient,.

Based on these curves and the knowledge of the river bottom composition, as well as of the morphologic characteristics of the watercourse, the curves of dependence of the Manning's coefficient upon discharge were defined for each river section. They will be used as the initial values of the parameters in the calibration process. The calibration process modifies these curves in order to achieve the minimum deviation of simulated values from the respective measurements.

For the comparison of calculated and measured values of water level the corresponding error norm was used. In accordance with this, the values that result in the minimum value of error norm were selected as the optimum values of Manning's coefficient. The purpose of this calibration scheme is the determination of the values of parameters within the range of acceptable solutions, so that the objective function has its minimum value.

The square root of the sum of squared deviations was adopted as the error norm. This function is often used for model calibration, but it uses the squared differences as the measure of deviation. Consequently, the difference of 10 cm effectively represents a deviation that is 100 times greater than the 1 cm difference.

Error norm has the following form:

$$J = \frac{1}{N_z} \sqrt{\sum_{1}^{N_Z} \left[ z_{M_i} - z_{S_i} \right]^2} \tag{22}$$

where :

$J$ – standard error,

$N_z$ – number of calculated ordinates of the level-gram,

$z_{M_i}$ - measured level and

$z_{S_i}$ - calculated level.

Genetic algorithms were applied to the calibration of the model, i.e. to the estimation of Manning's coefficient (as a function of discharge). In this concrete case of calibration of Manning's coefficient, one proposed set of coefficients represents an individual. The proposed set of Manning's coefficient values is used for the simulation of all selected periods. After the evaluation of individuals in the population, the further steps of the genetic algorithm are applied.

## 8. Results

Several periods different from the period used have been selected for the calibration procedure for verification of the parameters of the simulation model. The following figures show the graphic comparisons of observed and simulated values for the representative period from August 21st, 2006 to August 28th, 2006.

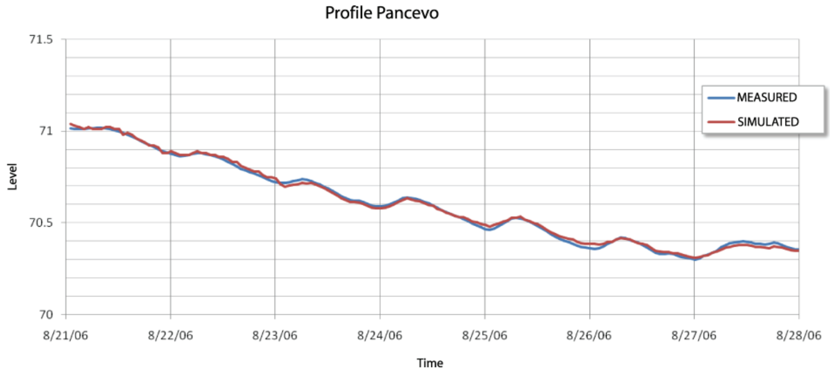


**Fig. 17.** Comparison of observed and simulated values of water levels on the control profile "Pančevo".

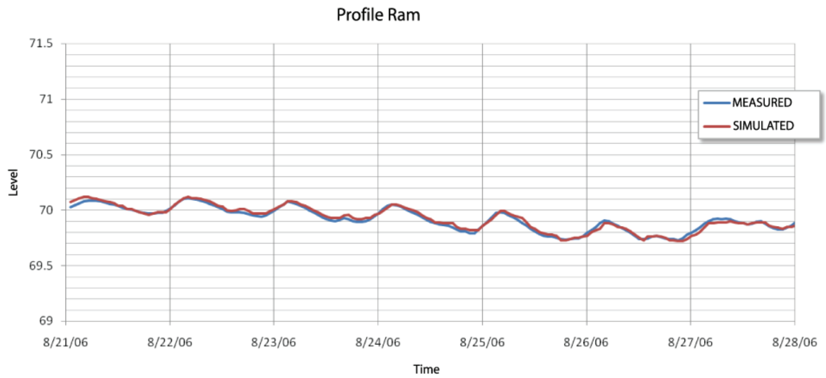


**Fig. 18.** Comparison of observed and simulated values of water levels on the control profile "Ram".

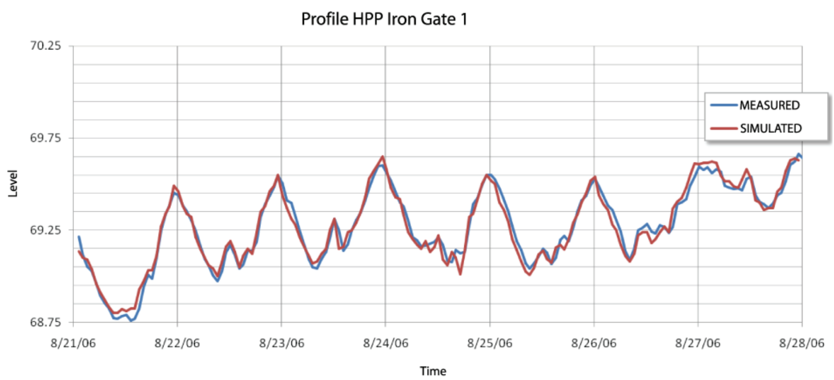


**Fig. 19.** Comparison of observed and simulated values of headwater level of "Iron Gate 1" HPP.

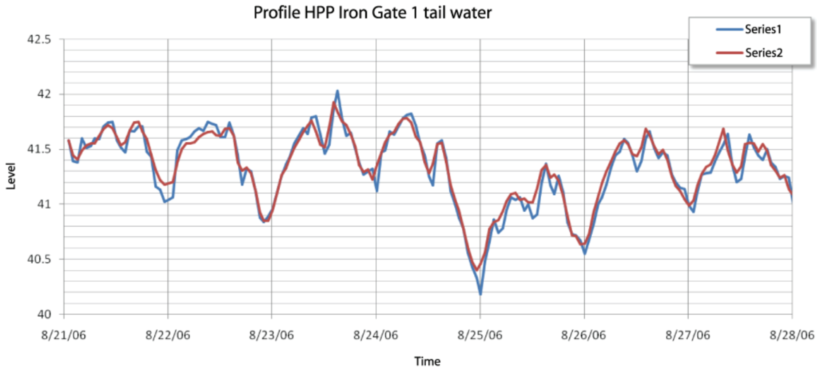


**Fig. 20.** Comparison of observed and simulated values of tailwater level of “Iron Gate 1” HPP.

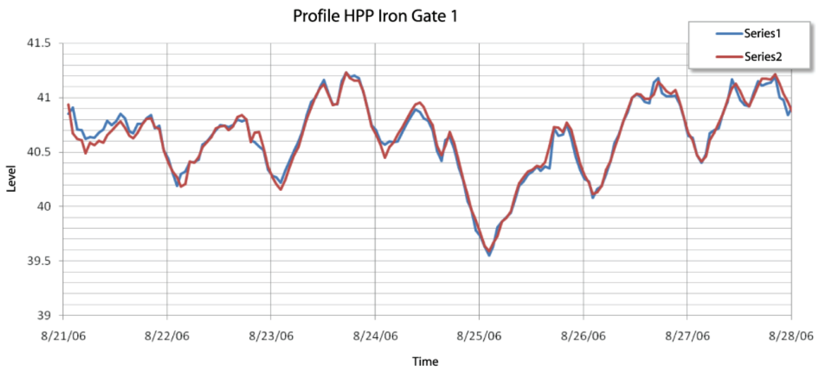


**Fig. 21.** Comparison of observed and simulated values of headwater water levels of “Iron Gate 2” HPP.

The model is calibrated and verified in the range of total inflow into the storage from 3000 and 8500 $m^3/s$. The periods of quasi-stationary inflows, as well as with the events of sudden changes in inflow, both in terms of its increase and its decrease are covered in this period. Beyond these boundaries, the model uses the extrapolated curves and the results are not of the same accuracy as when the data from the subjected interval is used. Presented results indicate that the model has been calibrated up to the level where the effect of the error in estimates is sufficiently low to exert no major impact on accuracy of the results obtained from the operation of hydropower objects.

## 9. Conclusion

This paper presents the numeric model of unsteady open channel flows with taking into account of the interaction of the flow with hydropower plants. The level of model complexity provides for taking into account of all specific points of the real system. The primary application of the model is its use in the systems that provide support to decision-making at the level of a dispatcher, but at the management level as well. Theoretical background of the models allows for the introduction of new elements into the existing electricity generation objects, as well as the introduction of completely new objects. Simulation model belongs to the group of models which can be used for short-term, as well as for medium-term and long-term calculations applied in planning of electricity generation, evaluation of the impact on the environment and the riparian areas, determination of rules of exploitation when the subject hydropower potential is shared by several entitles etc.

Since the simulation of the hydropower parts of the system parts is coupled with the flow simulation, there are no limitations regarding model application, i.e. there are no simplifications that would make the application of a certain model configuration impossible. Therefore, all types of turbines and management algorithms can be integrated in the "internal" models of the hydropower systems, and, in terms of flow, the storages of any size, complex flow networks and similar can be the subjects of this analysis.

The model was applied to the system for support to management of one of the biggest hydropower systems in Europe – the "Iron Gate" HPP system. Upon the results of model calibration and the results obtained by testing performed by the dispatching department of the "Iron Gate" HPP, it can be stated that the model has fulfilled the defined demands. Further model testing in everyday operation of the "Iron Gate" HPP dispatching department may point to eventual limiting factors in the model, as well as to the directions of its further development.

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