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Static and dynamic analysis of inelastic solids and structures by the BEM

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Abstract:

This review paper describes the techniques, advances and problems associated with the static and dynamic analysis of inelastic solids and structures by the Boundary Element Method (BEM). Firstly, an historical overview is presented. Next, the various existing BEM formulations for static and dynamic analysis of two- and three-dimensional solids and structures as well as plates and shells are briefly described. Inelasticity refers to elastoplastic, damage or elastoplastic plus damage material behavior. Then, eight characteristic numerical examples are presented to illustrate the methods and demonstrate their capabilities as well as their accuracy. Finally, advantages and disadvantages of the various methods as well as future developments are presented in the conclusions.

Key words: Boundary element method, damage, plasticity, static analysis, dynamic analysis

1. Introduction

During the last forty years or so, substantial advances have been made in the numerical analysis of basic structural elements as well as highly sophisticated structures exhibiting inelastic material behavior. These include, among others, many civil structures (e.g. buildings, bridges, tunnels, offshore platforms, tanks, chemical factories, power plants), or naval, aerospace and mechanical structures (e.g. trucks, vehicles, railways, ships, airplanes) involving inelastic behavior under extreme static or dynamic loads.

It should be noted that for the aforementioned realistic engineering problems, static or dynamic inelastic analysis is carried out exclusively by numerical methods due to their high complexity. In the last four decades, with the drastic evolution of digital computers, the finite element method (FEM) has assumed an important role in the solution of these complex engineering problems. The boundary element method (BEM) is about one decade younger for static and two decades younger for dynamic inelastic problems. Finite elements are the most frequently used numerical method today [1]. Even though the BEM can also be adopted to solve these problems with the same or better convenience and accuracy, it plays a secondary role in practical applications [2]. Even today, scientists and engineers find difficult to understand and program boundary element methods, mainly due to its focused presentation on mathematics.

On the other hand, FEM takes precedence not only due to its former origin but also due to the continuous appearance of specialized books and the existence of many powerful commercial programs, related to static and dynamic inelastic analysis.

The purpose of this review paper has to do with the critical presentation of the various existing BEM's for static and dynamic inelastic problems and the demonstration that these methods can represent a powerful alternative to the FEM in applications. Firstly, an historical overview is presented. Then, the various BEM's as applied to static and dynamic inelastic analysis of two- (2-D) and three-dimensional (3-D) solids and structures as well as plates and shells are briefly presented and discussed. Eight characteristic examples from the literature are presented to illustrate the applicability and accuracy of these boundary element methodologies. The paper is completed with conclusions pertaining to the advantages and disadvantages of the presented methods and to future developments.

2. Historical overview

2.1 BEM's for static inelastic analysis

In this section, the most significant points for the static and dynamic inelastic BEM history are presented. The presently available BEM's for static inelastic analysis are divided into two major categories: two- and three-dimensional solids and structures, and structures consisting of various structural members, such as beams, plates and shells.

2.1.1 Static inelastic analysis of 2-D and 3-D solids and structures

Swedlow and Cruse [3] presented the first elastoplastic BEM formulation. Various difficulties, such as the strongly singular inelastic domain integrals and the stability of the system equations have hindered its development. Riccardella [4] implemented an algorithm for 2-D elastoplasticity using piecewise constant interpolation of plastic strain rates. In computing interior stresses, Riccardella [4] recognized the strongly singular nature of the volume integral involving plastic strains. Telles and Brebbia [5,6] and Telles [7] solved a variety of 2-D single region test problems involving strain hardening and perfect plasticity. Mukherjee and his co-workers [8–12] used the initial strain approach to establish the boundary element formulation of the inelastic equations. Banerjee and his co-workers [13,14] presented an initial stress formulation and showed examples of 2-D, axisymmetric as well as 3-D problems. Later, advanced formulations of the boundary element method was developed by Banerjee et al. [15] and Banerjee and Raveendra [16,17] for inelastic analysis based on earlier the initial stress approach. These formulations were extended to axisymmetric and 3-D problems by Henry and Banerjee [18,19] and Banerjee et al. [20]. Based on these formulations, Chopra and Dargush [21] derived an advanced Newton–Raphson algorithm for elastoplasticity while a similar approach was also presented in Gao and Davies [22, 23], Poon et al. [24], Bonnet and Mukherjee [25], Cisilino and Aliabadi [26] and Wang et al. [27].

BEM has also been effectively applied to elastoplastic contact analysis. One can mention here the works of Karami [28], Huesmann and Kuhn [29] and Aliabadi and Martin [30, 31] for two-dimensional problems and the works of Gun [32, 33] and Liu and Shen [34] for three-dimensional elastoplastic contact problems.

BEM's have also reached a rather mature level for the inelastic analysis of structures taking into account geometric nonlinearities , i.e. large displacements and large displacements/large strains. The reader can consult the works [35-45].

In order to examine localization phenomena in elastoplastic solids and structures, Maier et. al [46], Benallal et al. [47] and Gun and Becker [48] proposed advanced non-local and gradient plasticity BEM formulations.

There is a rather small number of structural applications analyzed using the promising theory of damage mechanics in the framework of the BEM. The first integral formulation using damage mechanics appears to be the one by Rajgelj et al. [49]. Herding and Kuhn [50] developed an elastoplastic-damage BEM formulation. Sellers and Napier [51] and Cerrolaza and Garcia [52] used damage models to solve by the BEM characteristic problems in geomechanics. Localization phenomena in damaged solids and structures were examined by Garcia et al. [53], Lin et al. [54], Sladek et al. [55] and Benallal et al. [56]. Finally, Hatzigeorgiou and Beskos [57] developed the first 3-D damage mechanics formulation in the framework of the BEM.

2.1.2 Static inelastic analysis of beams and plates

This category of BEM's involves static inelastic analysis of structural elements, such as beams and plates. Moshaiov and Vorus [58] presented the first BEM formulation for elastoplastic plate bending analysis considering Kirchhoff's theory. One can also mention here the work of Chueiri and Venturini [59] which deals with Kirchhoff's theory applied to the analysis of concrete slabs. The first BEM approach to analyze elastoplastic thick plates considering Reissner's theory was formulated by Karam and Telles [60]. Karam and Telles [61] and Ribeiro and Venturini [62] also worked on elastoplastic Reissner's plates by the BEM. Auatt and Karam [63] examined elastoplastic Reissner's plates by a multilayered approach. Finally, Supriyono and Aliabadi [64, 65] formulated BEM's for combined geometric and material nonlinearities for shear deformable plates where large deflection/small strain and elastic perfectly plastic material behavior are taken into account.

2.2 BEM's for dynamic inelastic analysis

BEM is also popular for the solution of inelastic dynamic problems involving two- and three-dimensional solids and structures, and structures consisting of other structural members, such as beams and plates, as it is evident in the review articles of Beskos [66,67] and Providakis and Beskos [68]. The presently available BEM's for inelastic analysis under dynamic loads can be divided into three major categories: two-dimensional solids and structures, three-dimensional solids and structures and structures consisting of other structural members, such as beams and plates.

2.2.1 Dynamic inelastic analysis of 2-D solids and structures

This category is related to two-dimensional structures under plane strain or plane stress case. This category has reached a rather mature level and six methodologies have already been developed. In the first one, the BEM in its direct conventional form and in conjunction with the elastostatic fundamental solution of the problem has been successfully used for the analysis of these problems. One can mention here the works of Carrer and Telles [69,70], Coda and Venturini [71], Soares et al. [72-74] using the Domain/Boundary Element Method (D/BEM) and treating both plastic stresses and inertial forces by internal cells. Hatzigeorgiou and Beskos [75] applied this approach to compute the seismic inelastic response of masonry bridges using damage mechanics to model inelasticity. In all these cases, a volume discretization is required for the whole structure due to inertia terms.

The second approach has to do with the adoption of time-dependent fundamental solutions of the problem. One can mention here the work of Telles et al. [76]. This approach presents the advantage of eliminating the inertial volume integrals and thus the domain discretization is restricted to those parts of the domain where plastic stresses are expected to develop. However, the method appears to be particularly complicated and time consuming because of the complex kernels involved and the need to satisfy causality at every time step [76]. Moreover, problems of stability may appear during the time integration process [77].

In the third one, the BEM in its direct conventional form and in conjunction with the elastostatic fundamental solution of the problem is formulated. However, the dual reciprocity technique (DR-BEM) is applied to transform the inertial volume integrals into surface integrals. Thus, the interior discretization with volume cells, due to inelasticity, is restricted only to regions expected to become inelastic. One can mention here the work of Kontoni and Beskos [78] and Czyz and Fedelinski [79].

The fourth methodology has to do with the BEM in its symmetric Galerkin form and in conjunction with the elastostatic fundamental solution of the problem. Generally, the symmetric Galerkin form is a non-traditional BEM formulation using the classical Betti's work reciprocity theorem with single-layer and double-layer sources, in such a way that the integral operator turns out to be symmetric with respect to a suitably defined bilinear form. The space discretization is achieved either on the basis of variational properties of the solution, or using a weighted-residual technique according to Galerkin's classical correlation between shape and weight functions. On can mention here the works of Frangi [80] and Frangi and Maier [81].

The fifth one deals with the hybrid BEM/FEM schemes in the time domain, which appropriately combine the advantages of both the FEM and the BEM. The finite element method, for instance, is well suited for materials with inelastic behavior. For this reason, the finite element discretization is applied to regions expected to become inelastic. On the other hand, for systems with infinite extension, which are expected to remain elastic, the use of the BEM in conjunction with the elastodynamic fundamental solution of the problem is by far more beneficial. Thus, the FEM/BEM coupling in the time domain has been successfully used to solve 2-D nonlinear dynamic soil/structure interaction problems where the inelastic structure and the surrounding soil part expected to become inelastic are simulated by the FEM and the remaining soil assumed to behave linearly by means of the BEM. One can mention here the works related to general 2-D structures by Pavlatos and Beskos [82] and Soares et al. [83], structures with reinforced media by Coda [84] underground structures by Adam [85] and Takemiya and Adam [86], earth dams by Abouseeda and Dakoulas [87], concrete gravity dams by Yazdchi et al. [88] and wall structures by von Estorff and Firuziaan [89].

Finally, the sixth methodology has to do with the combination of the aforementioned first and the second boundary element methods. More specifically, Soares et al. [90] appropriately divided the total domain into two sub-domains: one that behaves elastically and is modeled by the elastodynamic time-domain BEM formulation and the other that behaves inelastically and is modeled by the D/BEM.

2.2.2 Dynamic inelastic analysis of 3-D solids and structures

This category has to do with the dynamic inelastic analysis of 3-D solids and structures. Three basic methodologies have already been developed for this purpose. The first approach has to do with BEM using the elastostatic fundamental solutions of the problem (D/BEM). The analysis involves the determination of inelastic stresses and inertia terms, which as internal quantities, require an internal discretization with 3-D volume cells. One can mention here the works of Hatzigeorgiou and Beskos [91,92] for the analysis of general 3-D elastoplastic structures and Hatzigeorgiou and Beskos [93,94] for general 3-D damaged structures. Furtermore, Hatzigeorgiou [95], and Hatzigeorgiou and Beskos [96] developed appropriate D/BEM methodologies to solve dynamic inelastic three-dimensional soil/structure interaction problems for underground structures. It should be noted that the D/BEM presents the advantages of stability and low computational cost, which are essential for 3-D dynamic inelastic analyses.

In the second approach, the BEM in its direct conventional form and in conjunction with the elastodynamic fundamental solution of the problem has been successfully applied by Ahmad and Banerjee [97]. In this case, the internal discretization is applied only in those regions of the interior domain where the inelasticity is expected. However, as in the 2-D case, the method appears to be quite complicated and time consuming because of the complex kernels involved and the need to satisfy causality at every time step, while problems of stability may appear during the time integration process.

In the third one, the BEM in its direct conventional form and in conjunction with the elastodynamic fundamental solution of the problem has been successfully combined with the finite element method for the dynamic analysis of 3-D elastoplastic problems. This promising scheme seems to be immature in 3-D formulations and only the work of Firuziaan and von Estorff [98] can be mentioned here.

2.2.3 Dynamic inelastic analysis of beams and plates

This category is related to BEM dynamic inelastic analysis of structural elements. Two major subcategories exist. The first one is related to dynamic inelastic analysis of plates. Fotiu et al. [99] employed the elastodynamic fundamental solution of the problem in conjunction with modal synthesis to determine the dynamic response of viscoplastic damaging plates. This method is very efficient but is restricted to very simple geometries. Providakis and Beskos and their co-workers [100-116] developed general D/BEM techniques for Kirchhoff plates, taking into account uniform or mixed boundary conditions, the effect of corners and elastic foundations and the influence of internal support conditions. These techniques have been also extended by Providakis [107,108] and Providakis and Beskos [109] to thick (Reissner–Mindlin) plates where shear deformations are taken into account.

The second subcategory is related to dynamic elastoplastic analysis of beam structures. One can mention here the works of Adam [110] and Adam and Ziegler [111,112]. In these works Green's functions in conjunction with modal analysis are adopted to create special BEMs for dynamic elastoplastic analysis of beams.

2.3 Treatment of volume integrals

Due to inelasticity, and sometimes due to inertia terms (e.g. in D/BEM formulation), the analysis requires the determination of internal quantities as inelastic stresses. Many techniques have been developed to replacing or transforming domain integrals into boundary integrals. One can mention here the works of Partridge et al. [113], Nowak and Neves [114], Wen et al. [115], Ma et al. [116], Nicholson and Kassab [117] and Gao [118]. Recently, Ribeiro et al. [119] have analyzed static elasto-plastic problems without a domain discretization prior to the analysis. Plasticity is assumed to start from the boundary and the cells are generated from the boundary data automatically during the analysis.

3. BEM's formulations for inelastic problems

This section briefly presents selected BEM's formulations for the analysis of inelastic problems. Since static analysis can be considered as a special case of the general dynamic analysis, the presented formulations are restricted to the latter case. Two major categories are examined: two-dimensional solids and structures and three-dimensional solids and structures. For the dynamic inelastic behavior of plates, the reader can consult the review paper of Providakis and Beskos [68].

3.1 BEM's for dynamic inelastic analysis of 2-D solids and structures

Firstly, the Domain/Boundary Element Method (D/BEM) is presented. For a 2-D body with volume Ω and surface Γ, the Somigliana identity for the dynamic inelastic case is defined as

$$\begin{aligned} c_{ij} u_j(\xi,t) = \int_\Gamma u_{ij}^*(\xi,X) p_j(X,t) d\Gamma(X) - \int_\Gamma p_{ij}^*(\xi,X) u_j(X,t) d\Gamma(X) - \\ - \rho \int_\Omega u_{ij}^*(\xi,X) \ddot{u}_j(X,t) d\Omega(X) + C_i \end{aligned} \tag{1}$$

where

$$C_i = \int_\Omega \varepsilon_{jki}^*(\xi,X) \sigma_{jk}^p(X,t) d\Omega(X) \tag{2}$$

for formulation associated to the initial stress approach, and

$$C_i = \int_\Omega \sigma_{jki}^*(\xi,X) \varepsilon_{jk}^p(X,t) d\Omega(X) \tag{3}$$

associated to the initial strain approach. In the above, t is the time, ρ the constant mass density of the body and $u_{ij}^*(\xi,X)$, $p_{ij}^*(\xi,X)$, $\varepsilon_{jki}^*(\xi,X)$, and $\sigma_{jki}^*(\xi,X)$ are the fundamental solution components of the elastostatic problem representing the displacement, traction, strain and stress, respectively. Besides, u_j, $\ddot{u}_j$, p_j, σ^p_{jk} and ε^p_{jk} represent the displacements, accelerations, tractions, inelastic stresses and inelastic strains, respectively. Furthermore, c_{ij} is the usual free coefficient of elastostatic analysis where $c_{ij}=\delta_{ij}$ for any interior point and $c_{ij}=\delta_{ij}/2$ for any point on the smooth boundary with δ_{ij} being the Kronecker constant.

Equation (1) represents the equation of motion of the body in integral form, and is reduced to the static case for $\rho=0$. The elastostatic (Kelvin) 2-D fundamental solution is adopted with expressions for displacement, traction, strain and stress of the form

$$u_{ij}^*(\xi,X) = \frac{(3-4\nu)\, ln(1/r)\delta_{ij} + r_{,i} r_{,j}}{8\pi(1-\nu)G} \tag{4}$$

$$p_{ij}^*(\xi,X) = \frac{\left[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}\right]\frac{\partial r}{\partial n} + (1-2\nu)(r_{,j}n_i - r_{,i}n_j)}{4\pi(\nu-1)r} \tag{5}$$

$$\varepsilon_{jki}^*(\xi,X) = \frac{(1-2\nu)(r_{,k}\delta_{ij} + r_{,j}\delta_{ik}) - r_{,i}\delta_{jk} + 2r_{,i}r_{,j}r_{,k}}{8\pi(\nu-1)Gr} \tag{6}$$

$$\sigma^*_{jki}(\xi,X)=\frac{(1-2\nu)(r_{,k}\delta_{ij}+r_{,j}\delta_{ki}-r_{,i}\delta_{jk})+2r_{,i}r_{,j}r_{,k}}{4\pi(\nu-1)r^2} \tag{7}$$

In the above, n is the surface (boundary) normal vector and r and $r_{,i}$ is the distance and its derivative along the i-axis, respectively, between field point X and collocation point ξ.

The boundary of the 2-D body is discretized into NB boundary elements and the domain into NV volume cells. Adopting the initial stress formulation, Eq. (1) becomes

$$\begin{aligned} c_{ij}u_j(\xi,t)=&\sum_{m=1}^{NB}\left\{\int_{\Gamma m}u^*_{ij}(\xi,X)\Phi d\Gamma\right\}p_j(X,t)-\\ &\sum_{m=1}^{NB}\left\{\int_{\Gamma m}p^*_{ij}(\xi,X)\Phi d\Gamma\right\}u_j(X,t)-\\ &\rho\sum_{n=1}^{NV}\left\{\int_{\Omega n}u^*_{ij}(\xi,X)\Phi d\Omega\right\}\ddot{u}_j(X,t)+\\ &\sum_{n=1}^{NV}\left\{\int_{\Omega n}\varepsilon^*_{jki}(\xi,X)\Phi d\Omega\right\}\sigma^p_{jk}(X,t) \end{aligned} \tag{8}$$

where Φ is the matrix of the shape functions. The boundary element implementation transforms the system of integral equations to an equivalent algebraic system, which in matrix notation reads

$$[G]\{p(t)\}-[H]\{u(t)\}-[M]\{\ddot{u}(t)\}+[Q]\{\sigma^p(t)\}=\{0\} \tag{9}$$

In the above, matrices $[G]$ and $[H]$ correspond to the boundary integrals and $[M]$ and $[Q]$ to the inertial and initial stress domain integrals, respectively. In this case, the volume discretization is required for the whole structure.

The second approach has to do with the adoption of time-dependent fundamental solutions of the problem. As it is expected, the analysis involves the determination of inelastic stresses, which as internal quantities, require an internal discretization. However, the internal discretization is applied only in those regions of the interior domain where inelasticity is expected. Equation (1) can also be used to express the integral equation of motion using initial stress or initial strain approach and the elastodynamic fundamental solution, under zero initial conditions and zero body forces. In this case, ξ and X correspond to field and source point, respectively. Furthermore, the independent variables ξ,X of u^*_{ij}, p^*_{ij}, ε^*_{ikj} and σ^*_{ikj} in Eq. (1) are replaced by ξ,τ; X,T and multiplications by time convolutions (*) as shown below:

$$u^*_{ij}*p_j=\int_0^T u^*_{ij}(X,T;\xi,\tau)p_j(X,\tau)d\tau \tag{10}$$

$$p^*_{ij}*u_j=\int_0^T p^*_{ij}(X,T;\xi,\tau)u_j(X,\tau)d\tau \tag{11}$$

$$\varepsilon^*_{jki}*\sigma^p_{jk}=\int_0^T \varepsilon^*_{jki}(X,T;\xi,\tau)\sigma^p_{jk}(X,\tau)d\tau \tag{12}$$

$$\sigma^*_{jki}*\varepsilon^p_{jk}=\int_0^T \sigma^*_{jki}(X,T;\xi,\tau)\varepsilon^p_{jk}(X,\tau)d\tau \tag{13}$$

Stresses at interior points are derived from displacements or from integral equations. According to the former approach, displacements lead to strains and strains to stresses through appropriate constitutive relations. This procedure is more computationally efficient. On the other hand, using the second approach, the integral equations for the stress state can be derived in the form

$$\begin{aligned}\delta\sigma_{jk}(\xi,T) = &\int_\Gamma \overline{u}^*_{ijk}(\xi,X,T) * \delta p_i(X,T)d\Gamma(X) - \\ &- \int_\Gamma \overline{p}^*_{ijk}(\xi,X,T) * \delta u_i(X,T)d\Gamma(X) + \\ &+ \int_\Omega \overline{\varepsilon}^*_{imjk}(\xi,X) * \delta\sigma^p_{im}(X,T)d\Omega(X) + \\ &+ R_{imjk}\delta\sigma^p_{im}(\xi,T)\end{aligned} \tag{14}$$

The knowledge of the response at time T_N requires the discretization of the time axis into N equal time intervals, i.e.,

$$T_N = \sum_{n=1}^{N} n\Delta T \tag{15}$$

Application of Eqs (10)-(13) and (15) into Eq. (1) gives the equation of motion of the structure in the form

$$\begin{aligned}c_{ij}(\xi)\Delta u_j(\xi,T_N) = &\int_{T_0}^{T_{N-1}}\int_\Gamma u^*_{ij}\Delta p_j d\Gamma d\tau + \int_{T_{N-1}}^{T_N}\int_\Gamma u^*_{ij}\Delta p_j d\Gamma d\tau - \\ &- \int_{T_0}^{T_{N-1}}\int_\Gamma p^*_{ij}\Delta u_j d\Gamma d\tau - \int_{T_{N-1}}^{T_N}\int_\Gamma p^*_{ij}\Delta u_j d\Gamma d\tau + \\ &+ \int_{T_0}^{T_{N-1}}\int_\Omega \varepsilon^*_{ikj}\Delta\sigma^p_{ik} d\Omega d\tau + \int_{T_{N-1}}^{T_N}\int_\Omega \varepsilon^*_{ikj}\Delta\sigma^p_{ik} d\Omega d\tau\end{aligned} \tag{16}$$

One must assume a variation of the field variables (i.e. displacements, tractions and stresses) during a time step. The simplest one is a constant variation during a time step. A linear variation gives

$$u_j(X,\tau) = \sum_{n=1}^{N} M^n_I u^{n-1}_j(X) + M^n_F u^n_j(X) \tag{17}$$

$$p_j(X,\tau) = \sum_{n=1}^{N} M^n_I p^{n-1}_j(X) + M^n_F p^n_j(X) \tag{18}$$

$$\sigma^p_{ik}(X,\tau) = \sum_{n=1}^{N} M^n_I \left(\sigma^p_{ik}\right)^{n-1}(X) + M^n_F \left(\sigma^p_{ik}\right)^n(X) \tag{19}$$

In the above, M_I and M_F are temporal interpolation functions of the form

$$M^n_I = \frac{T_n - \tau}{\Delta T}\varphi_n(\tau) \tag{20}$$

$$M_F^n = \frac{\tau - T_{n-1}}{\Delta T}\varphi_n(\tau) \tag{21}$$

where

$$\varphi_n(\tau) = H(\tau - (n-1)\Delta T) - H(\tau - n\Delta T) \tag{22}$$

with H being the Heaviside function. After the usual time and spatial discretization and integrations, the integral equation of motion (Eq. (16)) is transformed into an equivalent system of matrix equations of the form

$$\begin{aligned}&\left[A_F^1\right]\left\{\Delta X^N\right\} - \left[B_F^1\right]\left\{\Delta Y^N\right\} - \left[C_F^1\right]\left\{\left(\Delta\sigma^p\right)^N\right\} = \\ &-\sum_{n=2}^{N}\left[A_F^n + A_I^{n-1}\right]\left\{\Delta X^{N-n+1}\right\} + \sum_{n=2}^{N}\left[B_F^n + B_I^{n-1}\right]\left\{\Delta Y^{N-n+1}\right\} - \sum_{n=2}^{N}\left[C_F^n + C_I^{n-1}\right]\left\{\Delta\left(\Delta\sigma^p\right)^{N-n+1}\right\}\end{aligned} \tag{23}$$

or

$$\left[A_F^1\right]\left\{\Delta X^N\right\} = \left[B_F^1\right]\left\{\Delta Y^N\right\} + \left[C_F^1\right]\left\{\left(\Delta\sigma^p\right)^N\right\} + \left\{R^N\right\} \tag{24}$$

Similarly, the integral equation for stresses can be written as

$$\left\{\Delta\sigma^N\right\} = \left[A_F^1\right]_\sigma\left\{\Delta X^N\right\} + \left[B_F^1\right]_\sigma\left\{\Delta Y^N\right\} + \left[C_F^1\right]_\sigma\left\{\left(\Delta\sigma^p\right)^N\right\} + \left\{R^N\right\}_\sigma \tag{25}$$

This approach presents the advantage of eliminating the inertial volume integrals and thus the domain discretization is restricted to those parts of the domain where plastic stresses are expected to develop. Nevertheless, the method appears to be particularly complicated and time consuming because of the complex kernels involved and the need to satisfy causality at every time step. Moreover, problems of stability may appear during the time integration process.

In the third approach, the BEM in its direct conventional form and in conjunction with the elastostatic fundamental solution of the problem is again formulated. However, the dual reciprocity technique (DR-BEM) is applied to transform the inertial volume integrals into surface integrals. More specifically, the inertial domain integral of Eq. 1 can be transformed into boundary integrals by approximating the accelerations $\ddot{u}_j$ within the domain. Thus, accelerations $\ddot{u}_i(X,t)$ can be expressed by a sum of m coordinate functions $f^k(X)$ multiplied by the unknown time dependent functions $a_i^k(t)$, i.e.,

$$\ddot{u}_i(X,t) = \ddot{a}_i^k(t) f^k(X) \tag{26}$$

with summation on $k = 1$ to m implied. Thus, the internal integral of Eq. (1) becomes

$$\begin{aligned}\int_\Omega u_{ij}^*(\xi,X)\ddot{u}_i(X,t)d\Omega(X) = \Big[c_{ij}(\xi)\psi_{il}^k(\xi) &+ \int_\Gamma p_{ij}^*(\xi,X)\psi_{il}^k(X)d\Gamma(X) \\ &- \int_\Gamma u_{ij}^*(\xi,X)\eta_{il}^k(X)d\Gamma(X)\Big]\ddot{a}_l^k(t)\end{aligned} \tag{27}$$

where ψ_{il}^k and η_{il}^k denote the displacements and tractions solutions, respectively, of the 'pseudo-state' static problem

$$\bar{\sigma}^{k}_{i\,ln,n} + \delta_{il} f^{k} = 0 \tag{28}$$

with $\bar{\sigma}^{k}_{i\,ln,n}$ being the stress tensor corresponding to the above displacements, ψ^{k}_{il}, and tractions, η^{k}_{il}.

3.2 BEM's for dynamic inelastic analysis of 3-D solids and structures

This category has to do with the dynamic inelastic analysis of 3-D solids and structures. The integral equation of motion using initial stress or strain approaches is also given by Eq. (1). The first 3-D formulation adopted time-dependent fundamental solutions of the problem [97]. However, the method suffers from the shortcomings already mentioned in Section 3.1 in connection with the 2-D case. Recently, considerable progress has been realized in connection with the dynamic inelastic analysis of 3-D structures using the D/BEM approach. The advantages of this approach have been clearly stated in Section 3.1 in connection with the 2-D case. The elastostatic 3-D fundamental solutions are given by

$$u^{*}_{ij}(\xi, X) = \frac{(3-4\nu)\delta_{ij} + r_{,i} r_{,j}}{16\pi(1-\nu)Gr} \tag{29}$$

$$p^{*}_{ij}(\xi, X) = \frac{\left[(1-2\nu)\delta_{ij} + 3r_{,i} r_{,j}\right]\dfrac{\partial r}{\partial n} + (1-2\nu)\left(r_{,j} n_{i} - r_{,i} n_{j}\right)}{8\pi(\nu-1)r^{2}} \tag{30}$$

$$\varepsilon^{*}_{jki}(\xi, X) = \frac{(1-2\nu)\left(r_{,k}\delta_{ij} + r_{,j}\delta_{ik}\right) - r_{,i}\delta_{jk} + 3r_{,i} r_{,j} r_{,k}}{16\pi(\nu-1)Gr^{2}} \tag{31}$$

Equation (9) is used again to determine the unknown quantities of the problem.

4. Numerical examples

This section presents four static and four dynamic representative numerical examples to illustrate the various BEM's described in this paper and demonstrate their capabilities and accuracy.

4.1 Shallow tunnels under static surface loading

Davies and Gao [23] examined the elastoplastic behavior of two parallel arch tunnels subjected to surface foundation loading. Figure 1 shows the geometry of the problem and the corresponding BEM discretization.

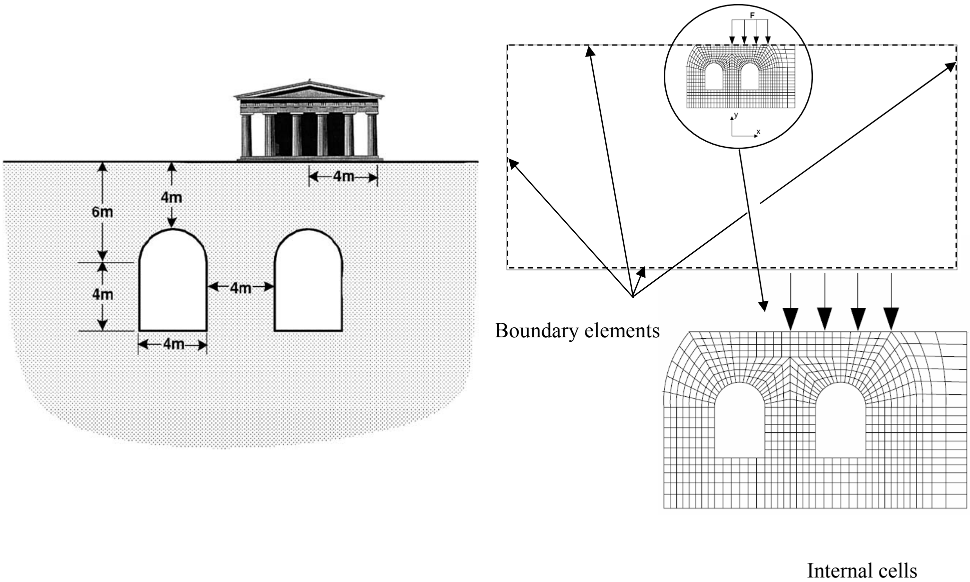

Fig. 1. Geometry and BEM discretization of the Example 4.1.

The physical properties of the examined model are: Poisson's ratio ν=0.22 and modulus of elasticity E=22·10^9 N/m^2 (competent rock). A perfectly plastic material obeying the Mohr–Coulomb yield criterion is assumed with cohesion c=1.3·10^6 N/m^2 and internal friction angle ϕ=31^o. Neither the in situ stress state nor the excavation process is taken into account. Figure 2 shows the deformed mesh under the influence of a foundation pressure of 4.2 MPa, with displacements magnified by factor of 50, and the yield nodes marked by small circles.

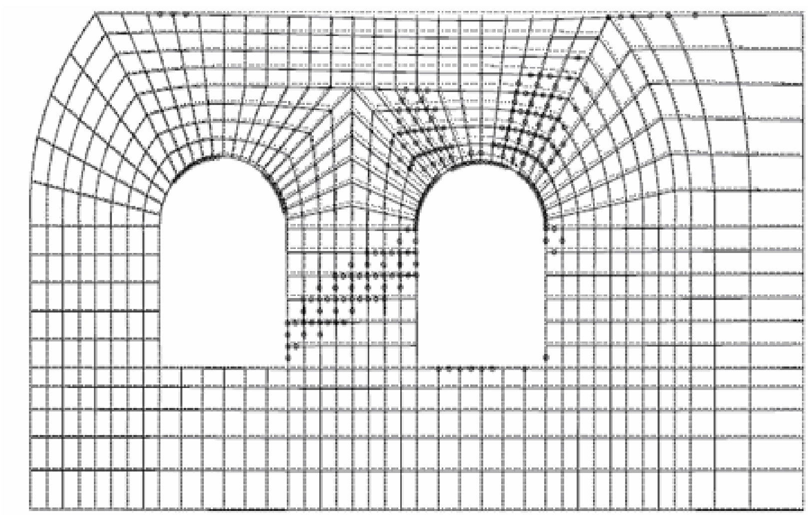

Fig. 2.. Deformed mesh and yielded nodes.

Finally, iso-stress contours of the second principal stress, σ_{II} (i.e., compressive stress) are shown in Fig. 3.

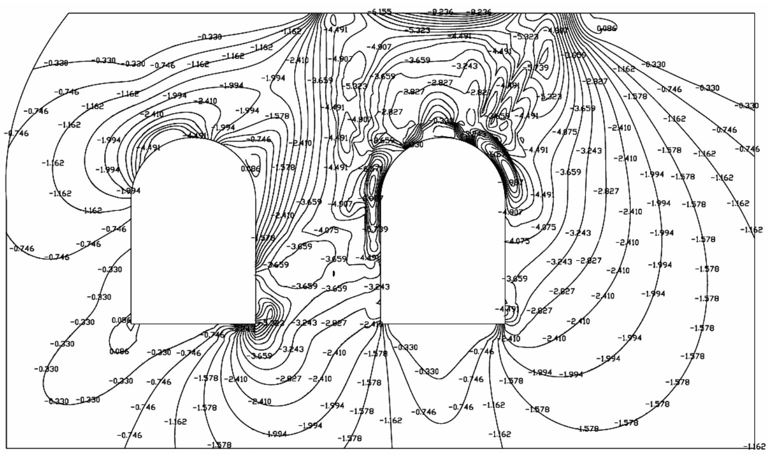

Fig. 3. Compressive stress contours.

4.2 A shear deformable plate with combined geometric and material nonlinearities

Supriyono and Aliabadi [65] examined a simply supported square plate, which is subjected to a uniform lateral distributed load q. The physical properties of the examined model are: Poisson's ratio ν=0.3, modulus of elasticity E=200·10^9 N/m^2 and yield stress σ= 300 MPa where the material is assumed to be an elastic perfectly plastic. The thickness to side ratio, h/a is equal to 0.05. Various meshes are used to study the influence of the discretization on the results. Figure 4 shows a typical mesh and the contour of plastic zone after plasticity takes place.

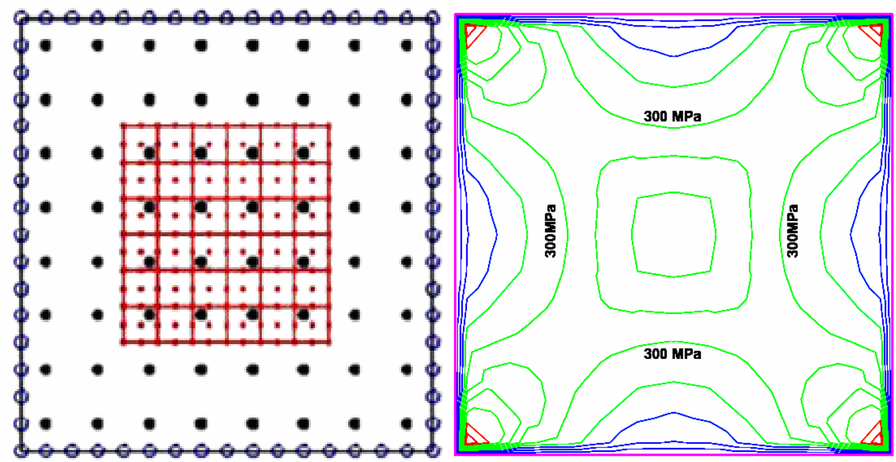

Fig. 4. Typical BEM mesh and iso-stress contours of plastic zones.

Figure 5 shows the normalized deflection *W* (deflection to plate thickness ratio, *w*/*h*) at the center point for various normalized load values $Q(=qa^4/Eh^4)$. The central deflection of the same square plate is also obtained by using of a commercial finite element method program.

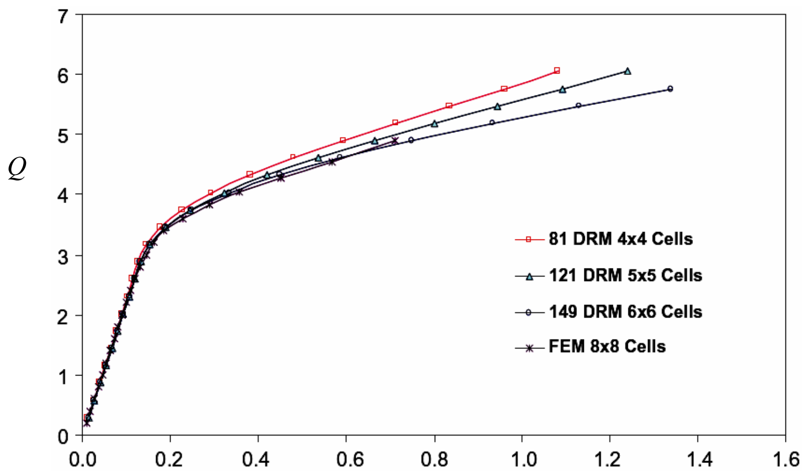

Fig. 5. Center point deflection computed by BEM and FEM.

4.3 Compression of a ceramic femoral head

Gun [32] examined the elastoplastic behavior of a ceramic femoral head, a 3-D structure with two contact regions involving three dissimilar materials. Figure 6 shows the details of the schematic diagram of the structural arrangement, the axisymmetric profile and the boundary element discretization of the titanium spigot, femoral head and brass ring.

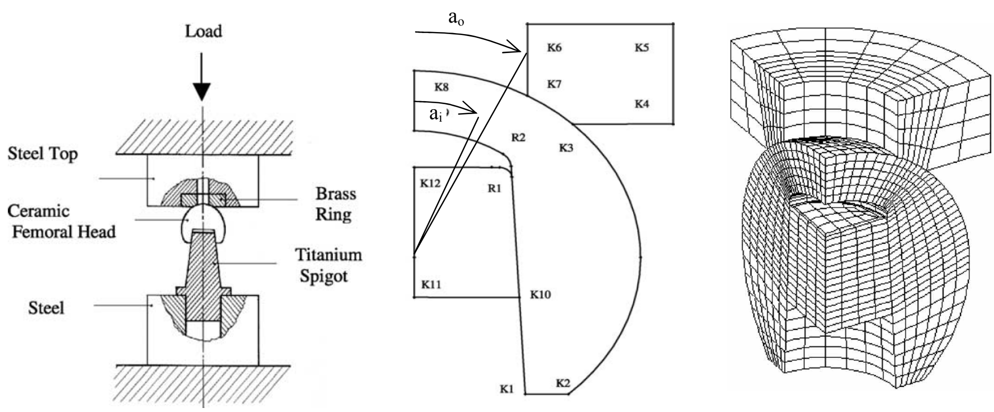

Fig. 6. Physical problem and BE mesh for femoral head assembly.

The titanium spigot is assumed to have the material properties: Young modulus E=1.14·10^{5}MPa, Poisson's ratio ν=0.33 and yield stress, σ=800 MPa. The femoral head is made of an alumina ceramic with E = 2·10^5 MPa, ν=0.3 and σ=200 MPa. A uniform displacement of 0.1 mm in vertical direction is applied to the brass ring, compressing the femoral head. The spigot is restrained in all directions between key points K10 and K11 and in the vertical direction between key points K11 and K12. The femoral head is restrained in the radial direction between key points K8 and K9. There are two distinct contact regions between the three parts; the contact region between the inside of the femoral head and the titanium spigot, which is assumed to be glued and the contact region between the outside of the femoral head and the ring, which is assumed to have slip–stick contact conditions. In the latter, three different values of coefficients of friction, μ = 0.0, 0.1 and 0.2 are employed in the analysis. Plastic deformations occur only in the femoral head. The von Mises stresses around the inside and the outside of the femoral head are shown in Fig.7.

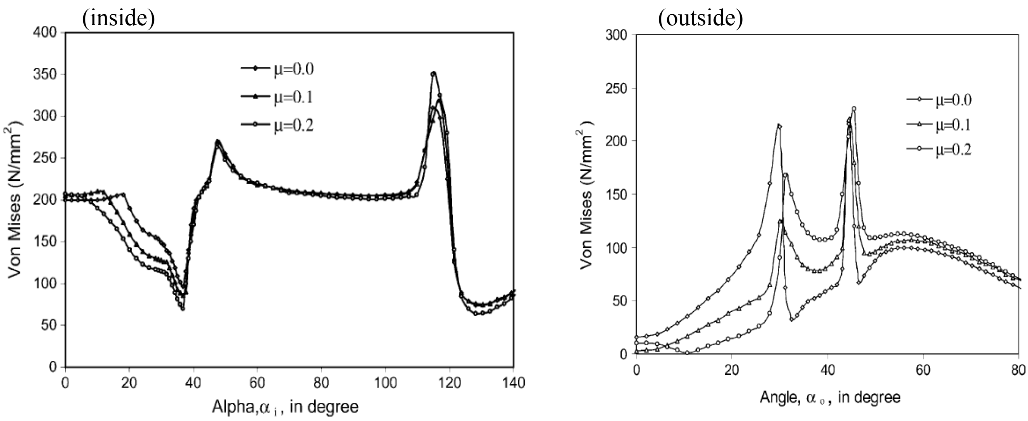

Fig. 7. Elasto-plastic stress distribution around the inside and outside of the femoral head.

4.4 Modeling of a cylinder-splitting test

In this example, a cylinder-splitting test is analyzed by Hatzigeorgiou and Beskos [57]. Cylinder splitting tests are frequently used to determine the tensile strength of concrete, rock and other quasi-brittle materials. The geometrical data appear in Fig. 8. This figure also contains the 3-D BEM discretization, where, due the symmetry, only one-eighth of the cylinder is discretized.

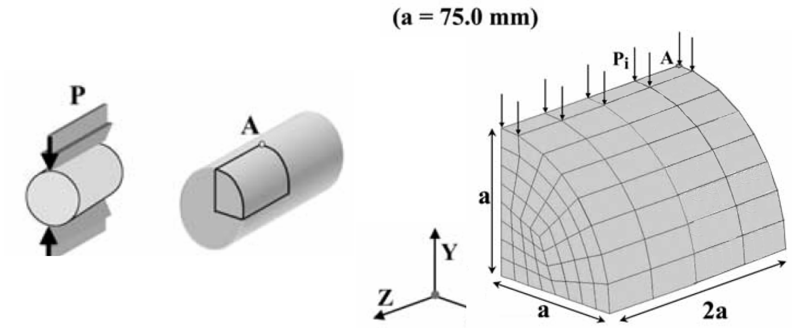

Fig. 8. Geometry and discretization of a cylinder-splitting test

The material parameters are: modulus of elasticity E=21.3 GPa, tensile strength of concrete f_t=1.7 MPa and Poisson's ratio v=0.15. The compressive strength is f_c=$f_{c(2\text{-}D)}$/1.15=21.3 MPa and the fracture energy G_f=50.0 N/m. It is worth noticing that the tensile strength results from the analytic relation $f_t = 2P_{max}/\pi LD$, where P_{max} is the maximum applied load and L and D are the length and diameter of the specimen, respectively. However, this tensile strength is considerably influenced by the boundary conditions, leading to scattered data. Upper and lower bounds for the ultimate load can be obtained by a limit analysis and variations of 25% from the mean value may be found. The above analytical relation for the given data provides an estimation of the maximum load, P_{max}=120.1 kN. The upper and lower bounds of the maximum load are found to be, P_{lower}=0.75 P_{max}=90.1 kN and P_{upper}=1.25 P_{max}=150.1 kN. The same problem has been worked out by Gomez and Awruch [120] using the FEM in conjunction with an elastoplastic model for concrete. The BEM and FEM results and the analytical maximum load P_{max} appear in the diagram of load–vertical displacement at point A of Fig. 9. It should be noted that BEM computes the maximum load as $P_{max(BEM)}$=1.034$P_{max(anal.)}$.

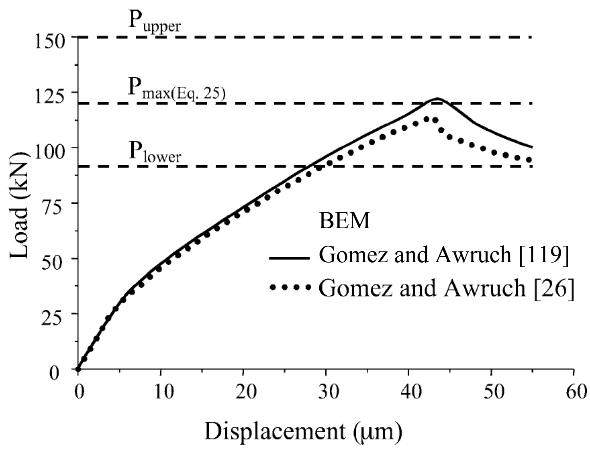

Fig. 9. Load-vertical displacement of point A diagram.

4.5 Dynamic analysis of an elastoplastic half-space

Soares et al. [83] examine an elastoplastic half-plane under a continuous, suddenly applied, stress distribution P_y=68.96 MPa along its surface, using a hybrid BEM/FEM scheme. The system is shown in Fig. 10a, where the geometry is defined by the distances a = 152.4m and b = 304.8m. Figure 10b shows the coupled BEM/FEM mesh where 90 linear boundary elements of equal length and 300 quadrilateral finite elements are used. The dimensions of the finite element subregion are c = 762m and d = 571.5m in horizontal and vertical direction, respectively.

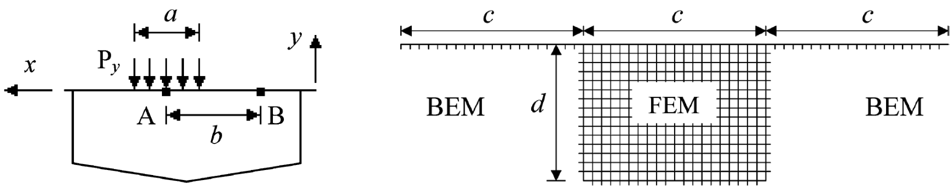

Fig. 10. Geometry and boundary conditions

In the FE sub-domain, a perfectly plastic material obeying the Mohr–Coulomb yield criterion is assumed. The material properties are: Poisson's ratio ν=0.25, Young modulus E=17.7 GPa, mass density ρ=31.5 kN·s^2/m^4, cohesion c=12.5 MPa and internal friction angle ϕ=10°. The time evolution of the plastic zone is shown in Fig. 11, which is obtained by the BEM/FEM scheme.

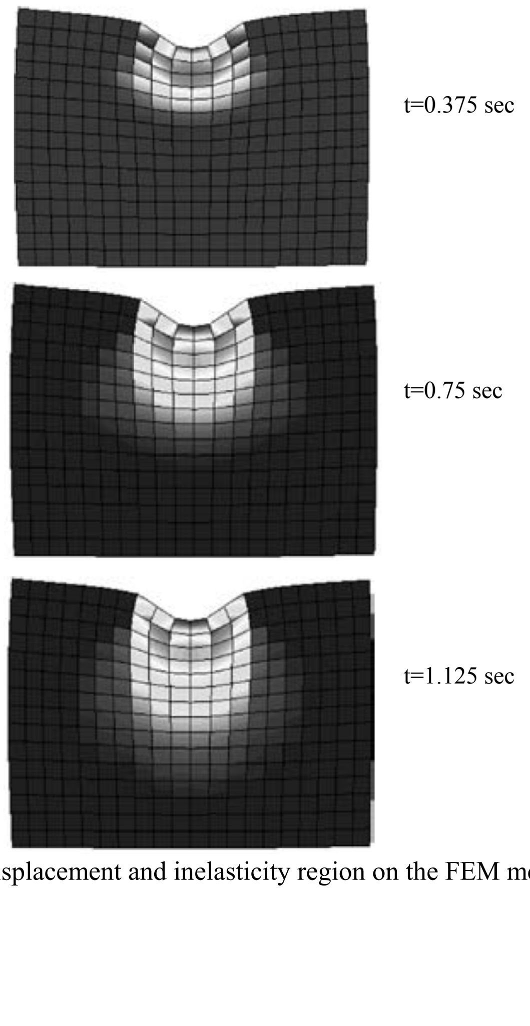

Fig. 11. Displacement and inelasticity region on the FEM mesh.

4.6 Dynamic analysis of an elastoplastic 3-D beam

In this example, a cantilever steel beam subjected to an impact loading P = 10 kN at its free end is analyzed numerically by Hatzigeorgiou and Beskos [92] using the three-dimensional D/BEM approach. The steel beam is simulated by the von Mises model of material behavior. Figure 12a contains the geometry and the 3-D BEM discretization of the structure, while Figs. 12b and 12c show the loading history and the stress–strain curve, which defines the material behavior. The material parameters are: Young modulus E=210.0 GPa, inelastic modulus E^T = 0.0 (i.e., elastic–perfectly plastic material behavior), Poisson's ratio ν=0.30, mass density ρ=7850 kg/m^3 and yield stress σ_y=400.0 MPa.

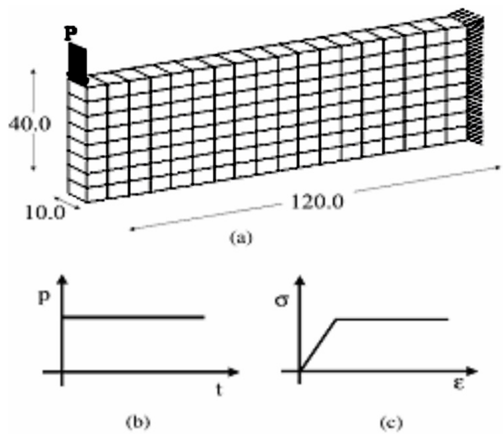

Fig. 12. (a) Geometry (in mm) and discretization,
(b) loading history and (c) material description.

Figure 13 depicts the elastic and inelastic time history of the vertical displacement at the load point, as computed by the D/BEM and the ANSYS finite element program.

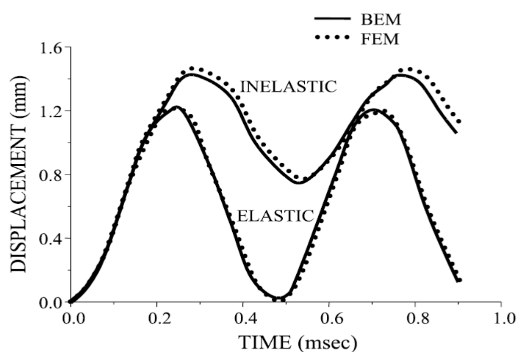

Fig. 13. Vertical displacement history at load point.

4.7 Dynamic analysis of a 3-D mortar beam

In this example, a simply supported mortar beam subjected to a central impact loading is analyzed numerically by Hatzigeorgiou and Beskos [94] using the three-dimensional D/BEM approach. The material is simulated by the FOM damage model, which has been proposed for concrete and other materials with quasi-brittle behavior. The assumed material parameters for the mortar beam are: Young modulus E=22·10^3 N/mm^2, Poisson's ratio ν=0.15, tensile strength f_t=3.91 N/mm^2, mass density ρ=2410 kg/m^3 and specific fracture energy G_f=103.7N/m. Figure 14a contains the geometry and the 3-D BEM discretization of the problem. Due to the symmetry, only half of the beam is discretized. Furthermore, Fig. 14b depicts the time history of the vertical concentrated load, which is applied at the upper face of the beam.

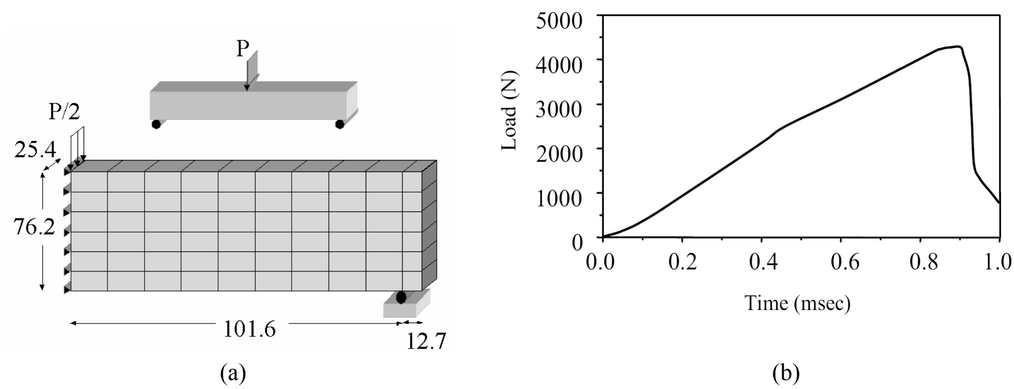

Fig. 14. a) Geometry (in mm) and boundary element discretization, b) Loading history.

The inelastic response of the beam has also been determined experimentally and numerically (using the finite element method) by Suaris and Shah [121]. Figure 15 shows the time history of the vertical displacement (deflection) at the load point, determined by the aforementioned numerical (FEM and D/BEM) and experimental approaches.

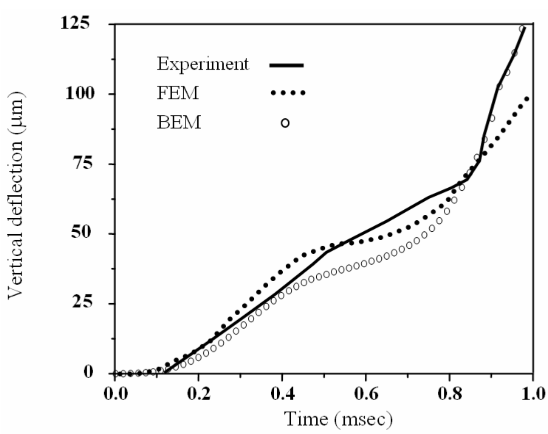

Fig. 15. Time history of deflection of load point.

4.8 Dynamic elastoplastic analysis of a thick elliptical plate

Providakis [107] examined a simply supported thick plate with elliptical boundary geometry under dynamic inelastic conditions. The two semi-axes of the elliptical shape are equal to 0.5 m and 0.6 m while the plate thickness is h = 0.15 m. This plate is subjected to a suddenly applied load uniformly distributed over its whole area with intensity 100 N/m^2 and rests on a Winkler-type foundation. Two different foundation rigidities are examined: k=0 and k=50·10^8 N/m. The material parameters for this example are: Young modulus E=200.0 GPa, inelastic (hardening) modulus E'=0.6E, Poisson's ratio ν= 0.3, yield stress σ_y=488.0MPa and mass density ρ=76900kg/m^3. The von Mises material model is adopted for the plate behavior. Figure 16 depicts the central deflection history of the elliptical plate as obtained by the D/BEM approach [107] in conjunction with 16 boundary and 64 interior elements per quadrant. The analysis time step is equal to Δt=10^{-6} sec. The central deflection of the same elliptical thick plate is also obtained by using NASTRAN finite element program and for the first foundation rigidity value (k = 0) and a mesh consisting of 40 solid finite elements in two layers. Figure 16 also shows the finite element analysis results, which are not as good as the ones by the D/BEM.

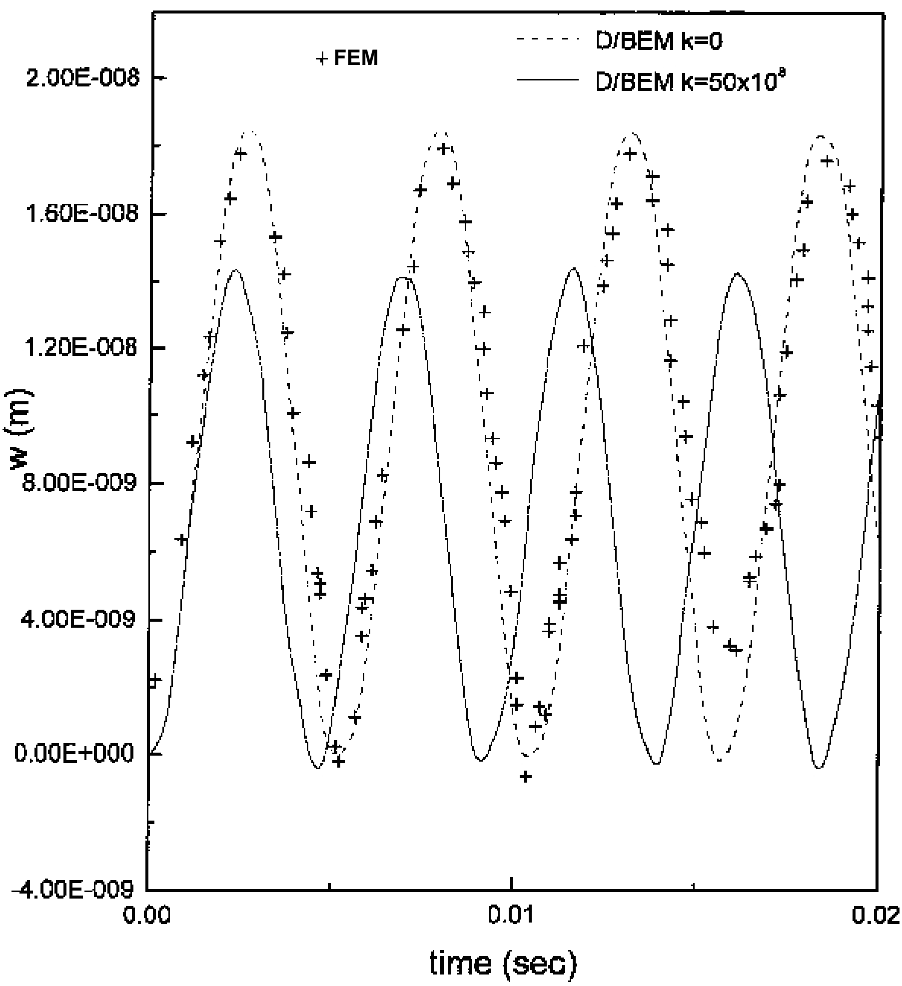

Fig. 16. Time history of vertical deflection of central point.

5. Conclusions and future research needs

This study leads to the following conclusions for static and dynamic inelastic analysis with BEM:

1) Numerous BEM's have been developed to solve static inelastic problems. Sections 2, 3 and 4.1-4.4 present the formulations and their capabilities. The use of the elastostatic fundamental solution requires a domain discretization, which, however, can be restricted to those parts of the domain expected to become inelastic. The current development for this topic seems to be mature.

2) Several BEM's have been developed to solve dynamic inelastic problems. Sections 2, 3 and 4.5-4.8 present the formulations and their potential. Among the existing BEM's for dynamic inelastic problems, the two most important approaches are those employing the elastodynamic and those employing the elastostatic fundamental solution. The first one has the advantage of restricting the interior discretization to those parts of the structure expected to become inelastic. The latter approach, eventhough requires a complete domain discretization, is considerably simpler and computationally more effective.

3) The BEM/D-BEM approach and the hybrid BEM/FEM approach seem to be very promising schemes for dynamic inelastic problems. These techniques combine the advantages of the concerned methods and are ideal for systems with limited inelastic regions as soil-structure interaction and fracture mechanics problems.

4) One major advantage of BEM for static and dynamic inelastic analysis has to do with the determination at once and for all times or loading steps of the concerned matrices. On the contrary, when using the FEM, the stiffness matrix is continuously varied and should be computed at every time or loading step. On the other hand, BEM matrices are full populated and non-symmetric (except for the symmetric Galerkin approach), while FEM matrices are symmetric and sparse and this affects efficiency considerably. However, the size of the BEM matrices is smaller than that of the FEM matrices.

5) Due to inelasticity, and sometimes due to inertia terms (e.g. in D/BEM formulation), the analysis requires the determination of internal quantities as inelastic stresses and hence an internal discretization. Thus, the main advantage of the BEM over the FEM of surface discretization is lost. However, as it is presented in Section 2.3, many techniques have been developed to avoid this drawback by replacing or transforming domain integrals into boundary integrals.

6) None commercial boundary element code has been developed for these problems. To the best of the authors' knowledge, the two existing general-purpose boundary element programs BEASY [128] and GPBEST [129] are not capable of treating inelastic behavior, both for static and dynamic problems.

Eventhough BEM's have reached a remarkable stage of development, there are still many areas where additional research is needed to be done in the future to further improve them and make them more competitive in practical applications. Among those, one can mention the following:

1) The basic two- and three-dimensional D/BEM formulations with damage should be extended to beams, plates and shells.

2) The promising hybrid BEM/FEM and BEM/D-BEM schemes should be extended to three-dimensions.

3) Appropriate extensions of the present 2-D and 3-D BEM's to the cases of non-local plasticity and non-local damage are required.

4) Much work is needed towards the development of BEM's for inelastic fracture mechanics problems. The corresponding boundary element formulations are limited either to linear dynamic or to nonlinear static analysis.

5) BEM's should be further extended to take into account large deformations or large strains/deformations in dynamic formulations.

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