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NUMERICAL SOLUTION OF NON-SIMILAR BOUNDARY-LAYER FLOW OVER A CYLINDER

Vahid Rezaee[1*] and Arash Houshmand[2]

[1] Department of Mechanical Engineering, Technical and Vocational University (TVU), Tehran, Iran
e-mail: vahid.rezaee@hotmail.com, vrezaee@tvu.ac.ir
[2] Department of Mechanical Engineering, Science and Research Branch, Islamic Azad University, Tehran (Markazi), Iran
e-mail: arash.houshmand@live.com
[*] *corresponding author*

Abstract

This paper presents a numerical solution of a non-similar two-dimensional boundary-layer flow over a cylinder. The assumptions and related two-dimensional flow equations are presented. The fourth-order Runge-Kutta method with the backward differentiation formula (BDF) method to separation point is implemented in the numerical solution using MATLAB software. Numerical solution results are compared with well-known analytic solutions. Shear stress diagram, friction ratio based on θ and x, and output results are illustrated. The results of numerical solution demonstrate good consistency with analytic solutions.

Keywords: Non-Similar Boundary-Layer Flow, Fourth-Order Runge–Kutta Method, Backward Differentiation Formula (BDF), Differential Method, Numerical Solution.

1. Introduction

In 1822, by means of the theory of continua, the complete equations of motion of viscous flows were introduced in famous Navier Stokes equations (Galdi 2000). The concept of the boundary layer in a fluid flow over a surface was presented by Ludwig Prandtl in 1904 (Anderson 2005). Prandtl in his famous lecture (Prandtl 1905) showed two regions can be defined for flow past a body. The first region, where the viscosity is important, is a thin transition layer close to the body called boundary layer. On the other hand, the second region, where the viscosity is important, is the remaining region outside the boundary layer (Schlichting and Gersten 2016). Considering small thickness of the boundary layer, certain approximations for the boundary layer can be applied such as the following: the variation of pressure normal to the wall is negligibly small and the variation of wall velocity is much smaller than normal velocity variation. Therefore, in two-dimensional flows x and y could be taken as the distances along and normal to the wall and u and v as the corresponding velocity components (Tani 1977).

The modern applications of boundary–layer theory are the calculation of friction drag of bodies e.g. a ship (Schlichting and Gersten 2016), behavior of turbulence (Mahrt 2014), Hypersonic boundary layer transition (Chen et al. 2021), nanofluids boundary layer flow (Awati, Goravar, and Wakif 2022; Gangadhar et al. 2020) and also magnetohydrodynamics (MHD) boundary layer flow of nanofluids analysis, especially regarding thermodynamic and heat transfer (Khashi'ie, Arifin, and Pop 2022; Rashid, Sagheer, and Hussain 2019; Reddy and Sreedevi 2021; Rashad, Chamkha, and Modather 2013). Similarity solutions of boundary layer flow are possible for some conditions (Aziz 2009). However, these new solutions remain valid for many cases (Ma and Hui 1990), even for non-Newtonian three-dimensional boundary layer equations (Timol and Kalthia 1986). In some conditions, for instance, when the velocity field becomes non-similar, the similarity equations could not be applied. Differential methods and integral methods are the main methods in non-similar solutions (Cebeci and Bradshaw 2012).Falkner and Skan presented boundary solution as the most widely used method among other solutions (White and Majdalani 2006; Falkneb and Skan 1931).

The similarity solutions are accurate but have limited applicability. So, when a higher degree of precision is needed, numerical models are suggested (White and Majdalani 2006). The greatest accuracy with the least computational cost is the main goal in the use of numerical methods. The accuracy of solution depends on the independent and dependent variables selection beside the used solution method (Schlichting and Gersten 2016). The Crank-Nicolson method (Crank and Nicolson 1947) and Keller's box method (Keller 1971) are the most commonly used numerical methods for boundary-layer equations. Among numerical methods for solving boundary-layer problems, the Box scheme method (Keller 1971), presented by Keller in 1971, obtained higher order accuracy (Keller 1978). Around the year 2000, it became possible to solve the full Navier-Stokes equations using the numerical methods in fluid mechanics. Reynolds Averaged Navier-Stokes were widely used in academic research and industry (Schlichting and Gersten 2016).

A numerical solution of non-similar boundary layer flow solution past a horizontal circular cylinder using the Brinkman model and Keller-box method was presented by Nazar et al. (Nazar et al. 2003). Skin friction, heat transfer and transverse curvature effect as numerical results were obtained by Datta et al. (Datta et al. 2006) in a non-uniform slot injection (suction) on a forced flow over a slender cylinder. A numerical parametric study was done by Rashad et al. (Rashad, Chamkha, and Modather 2013) for mixed convection boundary-layer flow past a horizontal circular cylinder embedded in a porous medium filled with a nanofluid. A numerical solution for non-similar boundary layer flow over a yawed cylinder using the implicit finite difference scheme and quasi-linearization technique was presented by Revathi et al. (Revathi, Saikrishnan, and Chamkha 2014). A numerical analysis using Keller Box method for double diffusive magnetohydrodynamic (MHD) transport phenomena for a nanofluid from a horizontal circular cylinder was presented by Ramachandra Prasad et al. (Prasad, gaffar, and Kumar 2019). A numerical solution of non-similar boundary layer flow of Sisko fluid over a stretching cylinder was evaluated by Cui et al. (Cui et al. 2021). A numerical solution for boundary layer flow over a nonlinear stretching surface with uniform lateral mass flux was provided by Afridi et al. (Afridi et al. 2022). When the computation of higher derivatives in solving differential equations is complicated, fourth order Runge-Kutta method is the most reliable conventional method (Islam 2015; Butcher 2016), especially in fluid mechanics applications (Carpenter et al. 2005). BDF–Runge–Kutta methods were implemented in numerical solutions in many cases (Vigo-Aguiar, Martín-Vaquero, and Ramos 2008; Ramos and Vigo-Aguiar 2007). In this paper, a numerical solution of a non-similar two-dimensional boundary-layer flow over a cylinder has been conducted using the fourth-order Runge-Kutta method and BDF method.

2. Model assumptions

The following assumptions about the flow over a cylinder are considered:

(a) The non-similar boundary-layer of two-dimensional laminar flow shown in Fig. 1 is assumed.
(b) $\theta = \frac{x}{r_0}$
(c) $Re < 10^4 < Re_{cr}$
(d) $u_e = 2u_\infty \sin\theta$
(e) $x = 0 \quad \rightarrow \quad x_{sep}$
(f) $\lambda_{sep} = -0.09$
(g) $\nu = 15.7 \times 10^{-6} \quad m^2/s$

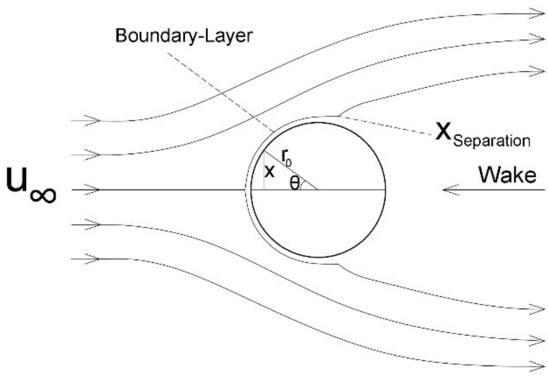

Fig. 1. The assumed Non-Similar Boundary-Layer flow.

3. Non-similar two-dimensional flow equations

The equations of flow motion with only two velocity components, u (x, y, t) and υ (x, y, t), considering the continuity and the Navier–Stokes equations are defined as (White and Majdalani 2006; Oleinik and Samokhin 2018):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \nu\frac{\partial u}{\partial y} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + \nu\frac{\partial v}{\partial y} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

where, x, y: Cartesian coordinates, u, v: Cartesian velocity components, t: Time, g: Gravitational acceleration vector, p:Pressure, ρ: density.

By the assumption of constant ρ, the stream function $\Psi\ (x, y, t)$ could be defined as in the following equations:

$$u = \frac{\partial \Psi}{\partial y} \tag{4}$$

$$v = -\frac{\partial \Psi}{\partial x} \tag{5}$$

A fourth-order partial differential equation can be achieved by taking the curl of vectorial momentum equation (White and Majdalani 2006):

$$\frac{\partial}{\partial \mathrm{t}}(\nabla^2 \Psi) + \frac{\partial \Psi}{\partial \mathrm{y}} \frac{\partial}{\partial \mathrm{x}}(\nabla^2 \Psi) - \frac{\partial \Psi}{\partial \mathrm{x}} \frac{\partial}{\partial \mathrm{y}}(\nabla^2 \Psi) = \nu \nabla^4 \Psi \tag{6}$$

Non-similar two-dimensional flow is defined when there is no change in external velocity distribution as in the following equation:

$$u_e = C x^m \tag{7}$$

where, u_e: External flow velocity, x: Longitudinal coordinate, C: a constant.

Considering equation (7), non-similar two-dimensional flow is defined when m is not a constant or if m is constant in a mass transfer stream but dimensionless stream function in wall (f_w) is not constant, as in the following equation(Cebeci and Bradshaw 2012):

$$f = -\frac{1}{(u_e \mathrm{v x})^{1/2}} \int_0^x \mathrm{v}_w dx, \qquad \text{Conditions: } \eta = 0,\ f' = 0,\ g = 0 \tag{8}$$

where, g: The dimensionless temperature difference, η: A similarity variable (special function of x and y) and it is defined as:

$$\eta = \frac{\mathrm{y}}{\delta(\mathrm{x})} \tag{9}$$

where, δ: The shear-layer thickness(mm).

4. Non-similar two-dimensional flow solutions

Non-similar solutions are based on similarity transformations. Non-similar flows can be solved by differential methods or integral methods. Differential methods are based on solving the partial differential equations for conservation of mass, momentum and energy. Integral methods are based on ordinary differential equations with x as the independent variable. Integral methods involve approximate data-correlation formulas (Cebeci and Bradshaw 2012).

4.1 Differential method

Falkner-Skan transformation for external boundary-layer flows is expressed as (Cebeci and Bradshaw 2012):

$$\Psi = (u_e \nu x)^{\frac{1}{2}} f(\eta) \tag{10}$$

$$\eta = (\frac{u_e}{\nu x})^{\frac{1}{2}} y \tag{11}$$

By allowing f to vary with x also, non-similar two-dimensional uncoupled flows, Falkner-Skan transformation is defined as:

$$\Psi = (u_e \nu x)^{\frac{1}{2}} f(x, \eta) \tag{12}$$

For a laminar uncoupled thin shear layer with δ/x or $d\delta/\mathrm{dx} \leqslant 1$, the x component momentum equation can be expressed as (Cebeci and Bradshaw 2012):

$$u \frac{\partial u}{\partial \mathrm{x}} + \nu \frac{\partial u}{\partial \mathrm{y}} = -\frac{1}{\rho} \frac{\partial \mathrm{p}}{\partial \mathrm{x}} + \nu \left(\frac{\partial^2 \mathrm{u}}{\partial y^2} + f_x \right) \tag{13}$$

Considering equations (4) and (5), stream function with the transformation defined by equations (11) and (12) together and with the replacement of the $-1/\rho\ dp/\mathrm{dx}$ by $u_e du_e/dx$ in equation (13), the transformed momentum and energy equations for two-dimensional uncoupled laminar flows can be expressed as (Cebeci and Bradshaw 2012):

$$f''' + \frac{m+1}{2} f f'' + m[1 - (f')^2] = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}\right) \tag{14}$$

where, m: A dimensionless pressure-gradient parameter defined as:

$$m = \frac{x}{u_e} - f'' \frac{du_e}{\mathrm{d}x} \tag{15}$$

Dimensionless stream function in the wall is the same as equation (8) with the following added conditions:

Added Conditions to Eq. (8): $\eta = \eta_e, \quad f' = 1, \quad g = 1$ (16)

where, η_e: transformed boundary-layer thickness.

4.2 Integral method

Pohlhausen's method (Pohlhausen 1921) and Thwaites' method (Thwaites and Meyer 1960) are the most frequently used methods among other Integral methods.

4.2.1 Pohlhausen's method

Due to impractical differential method solution of the boundary-layer equations in Pohlhausen method, this method was the most sophisticated method before general availability of computers. A velocity profile $u(x, y)$ that satisfies the momentum integral equation, i.e. Eq. (17), is assumed in this method. Also boundary conditions as $y = 0, u = 0, y \to \infty$ and $u = u_e(x)$ are considered (Cousteix and Cebeci 2005).

$$\frac{d\theta}{dx} + \frac{\theta}{u_e} \frac{du_e}{dx}(H + 2) = \frac{c_f}{2} \tag{17}$$

where, c_f: Local skin-friction coefficient. H: Shape factor and it is defined as:

$$H = \frac{\delta^*}{\theta} \tag{18}$$

where, δ^*: The displacement thickness of the boundary layer.

At the wall with $v_w = 0$, additional boundary conditions are obtained a s(Cebeci and Bradshaw 2012):

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho} \frac{\mathrm{dp}}{\mathrm{d}x} = -u_e \frac{du_e}{dx} \tag{19}$$

With respect to y, some more boundary conditions from differentiating the edge boundary conditions are:

$$y \to \delta \quad \text{then} \quad \frac{\partial u}{\partial \mathrm{y}}, \frac{\partial T}{\partial \mathrm{y}}, \frac{\partial^2 \mathrm{u}}{\partial y^2}, \frac{\partial^2 T}{\partial y^2}, \frac{\partial^3 \mathrm{u}}{\partial y^3}, \frac{\partial^3 T}{\partial y^3}, \ldots \to 0 \tag{20}$$

4.2.2 Thwaites' method

This method is used in calculating momentum transfer in uncoupled laminar flows with pressure gradient and without mass transfer. The Thwaites' dimensionless obtained differential equation is defined as (Cebeci and Bradshaw 2012):

$$\left(\frac{\theta}{L}\right)^2 = \frac{0.45}{(u_e^*)^6 R_L} \int_0^{x^*} u_e^{*5}\, dx^* + \left(\frac{\theta}{L}\right)_i^2 \left(\frac{u_{e_i}^*}{u_e^*}\right)^6 \tag{21}$$

where, L: reference length. The subscript i denotes the initial conditions at $x^* = 0$. Other dimensionless quantities are defined as:

$$x^* = \frac{x}{L} \tag{22}$$

$$u_e^* \equiv \frac{u_e}{u_{ref}} \tag{23}$$

$$R_L \equiv \frac{u_{ref}L}{\nu} \tag{24}$$

where, u_{ref}: reference velocity.

When θ is calculated, the H and c_f parameters can be determined as a function of Thwaites' pressure-gradient parameter $\lambda \equiv (\theta^2/\nu)du_e/dx$:

$$R_\theta \frac{c_f}{2} = 0.225 + 1.61\,\lambda - 3.75\lambda^2 + 5.24\lambda^3 \tag{25}$$

$$H = 2.61 - 3.75\lambda + 5.24\lambda^2 \tag{26}$$

For $-0.1 \leqslant \lambda \leqslant 0$ a suitable data fit is:

$$R_\theta \frac{c_f}{2} = 0.225 + 1.472\,\lambda + \frac{0.0147\,\lambda}{0.107 + \lambda} \tag{27}$$

$$H = 2.472 + \frac{0.0147}{0.107 + \lambda} \tag{28}$$

where:

$$R_\theta = \frac{u_e\theta}{\nu} \tag{29}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho u_e^2} \tag{30}$$

4.2.3 Integral solutions for two-dimensional non-similar flows

The variation of the dimensionless wall shear is shown in Fig. 2. The Thwaites' method result is shown by the solid line and similarity solution for a stagnation-point flow with m=1 is shown by the dashed line (based on Eq. (33)). Referring to Fig.2, the Thwaites' method result is in agreement with integral methods up to $\theta = 30^\circ$, because the flow is near the stagnation point. For small values of the angle θ (near stagnation point), $u_e$can be expressed as (Cebeci and Bradshaw 2012):

$$u_e = 2u_\infty\theta = \frac{2u_\infty x}{r_0} \tag{31}$$

Considering the definition of wall shear stress in the Falkner-Skan variables, Eq. (12) can be written as (Cebeci and Bradshaw 2012):

$$\tau_w = \mu\left(\frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right)_w = \mu u_e f_w^{''}\sqrt{\frac{u_e}{\nu x}} \tag{32}$$

By replacing Eq. (31) into Eq. (32) and considering $f_w^{''} = 1.23259$ for $m = 1$, Eq. (33) is obtained as:

$$\frac{\tau_w}{\rho u_\infty^2}\left(\frac{u_\infty r_0}{\nu}\right)^{1/2} = 2\sqrt{2}\theta \times 1.23259 = 3.486\,\theta \tag{33}$$

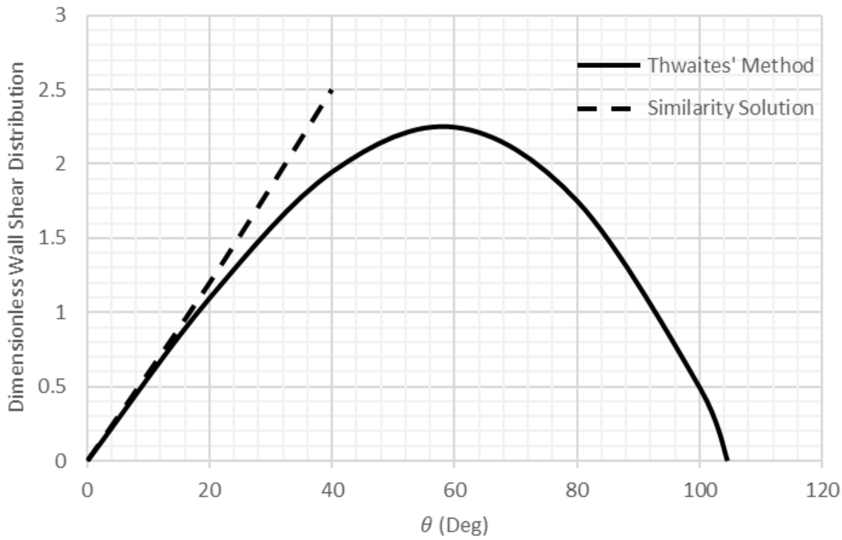

Fig. 2. The variation of dimensionless wall shear distribution $(\tau_w/\rho u_\infty^2)(u_\infty r_0/\nu)^{1/2}$ around the circumference of a circular cylinder (Cebeci and Bradshaw 2012).

4.3 Fourth-order Runge-Kutta method

In order to solve the initial value problems for ordinary differential equations, Runge-Kutta methods are the most efficient technique.

Ordinary differential equations form used for solving by Runge-Kutta Methods can be expressed as:

$$\frac{d_y}{d_x} = f\ (x, y) \tag{34}$$

The general form for this method can be written as Eq. (35):

$$y_{i+1} = y_i + \phi \mathrm{h} \tag{35}$$

where, ϕ: the slope estimate which is used to extrapolate from an old value y_i to a new value y_{i+1} over a distance h.

In Runge-Kutta methods, many variations exist but all can be written in the following generalized form (Chapra and Canale 2011):

$$y_{i+1} = y_i + \phi(x_i, y_i, h)\mathrm{h} \tag{36}$$

where $\phi(x_i, y_i, h)$: An increment function, can be defined as a representative slope over the interval and can be expressed as (Chapra and Canale 2011):

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n \tag{37}$$

where $a_1, \ldots, a_n$: constants and $k_1, \ldots, k_n$ are defined as (Chapra and Canale 2011):

$$
\begin{gathered}
k_1 = f(x_i, y_i) \\
k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\
\vdots \\
k_n = f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)
\end{gathered}
\tag{38}
$$

where: $p_1, \dots, p_n$ and $q_1, \dots, q_n$: constants. $k_1, \dots, k_n$: recurrence relationships.

Classical fourth order Runge-Kutta iterative method as a solution for differential equation $y'(x) = f(x, y)$ with initial condition $y(x_0) = y_0$ and with accumulative error in the order of $O(h^4)$ is developed as follows (Tan and Chen 2012):

$$
\begin{cases}
y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h, & \\
K_1 = f(x_i, y_i), & K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h), \\
K_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_2 h), & K_4 = f(x_i + h, y_i + K_3 h),
\end{cases}
\tag{39}
$$

Round-off errors and Truncation errors occurrences in numerical solutions of ordinary differential equations are inevitable. Rounding errors are generated by using fixed and limited numbers, and truncation errors happen due to approximations (Islam 2015).

The error accuracy can be increased by using higher order of Runge-Kutta methods, but calculation complexity should be also considered. Since the most efficient method for solving a numerical solution for initial value problems is the fourth order Runge-Kutta method, it is the most widely used technique by mathematicians (Tan and Chen 2012).

5. Numerical solution of non-similar boundary-layer flow over cylinder

The numerical solution, which was formulated for a non-similar two-dimensional Boundary-Layer flow over a cylinder case, is incorporated into MATLAB codes. Results are shown as diagrams and their analyses will be presented in the result and discussion section. The m-files (script file) code in MATLAB software which is used to solve equations is presented in Appendix I. Falkner-Skan equation by Fourth-order Runge-Kutta and differential equation with backward differentiation formula (BDF) method have been solved.

6. Results and discussion

The solution of non-similar two-dimensional Boundary-Layer flow over a cylinder using the fourth-order Runge-Kutta and backward differentiation formula (BDF) method with the use of MATLAB Software is presented in this section. The variation of the f , f' and f'' with η considering $m = -0.2$ is shown in Figs. 3,4 and 5.

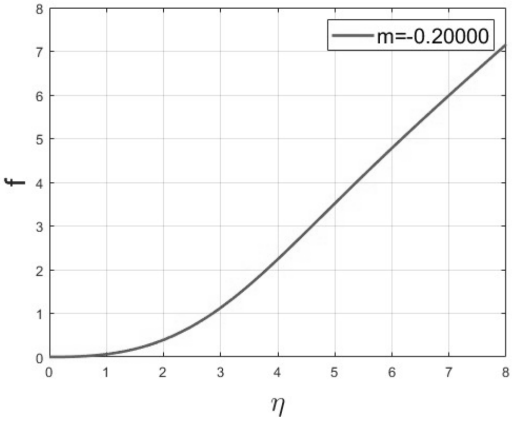

Fig. 3. Variation of f versus η.

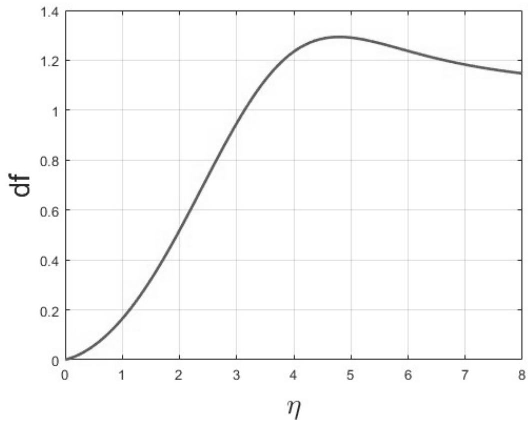

Fig. 4. Variation of f' versus η.

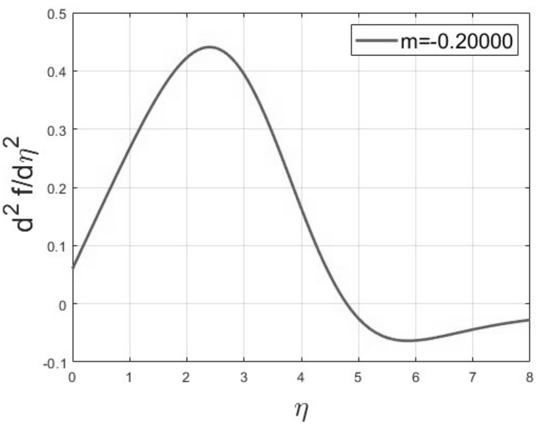

Fig. 5. Variation of f'' versus η.

Figure 6 depicts the variation of the friction coefficient (c_f) with θ angle. The friction coefficient (c_f) value decreases to a limit value when the θ angle increases.

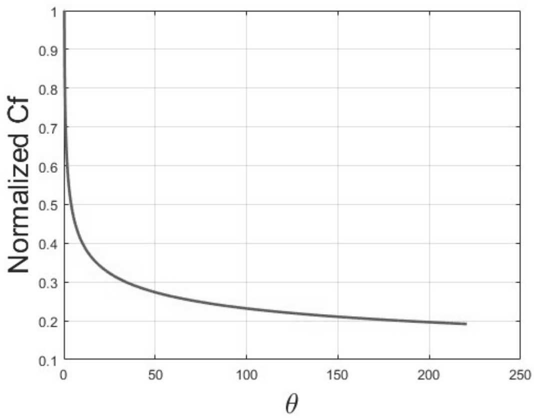

Fig. 6. Variation of c_f versus θ.

Figure 7 depicts the variation of the shear stress (τ) versus θ angle while Fig. 8 depicts the variation of the shear stress (τ) versus x. In both figures, shear stress reached the maximum value then it is decreased dramatically.

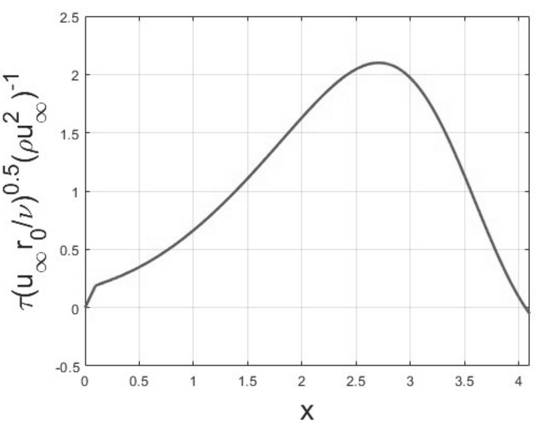

Fig. 7. Variation of shear stress (τ) versus x.

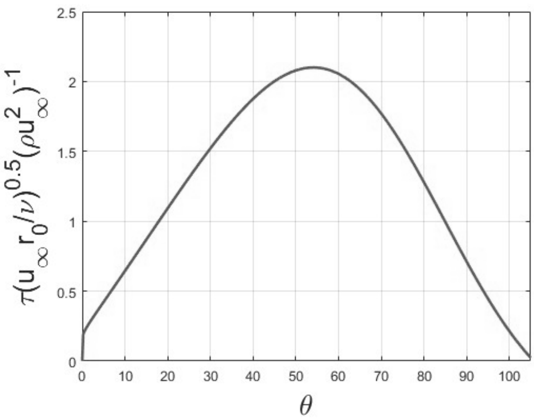

Fig.8. Variation of shear stress (τ) versus θ.

In order to ensure the accuracy of the model, the variation of the shear stress (τ) versus θ in the numerical solution in comparison with the obtained result with integral solution (Thwaites' method) and similarity solution is illustrated in Fig. 9.

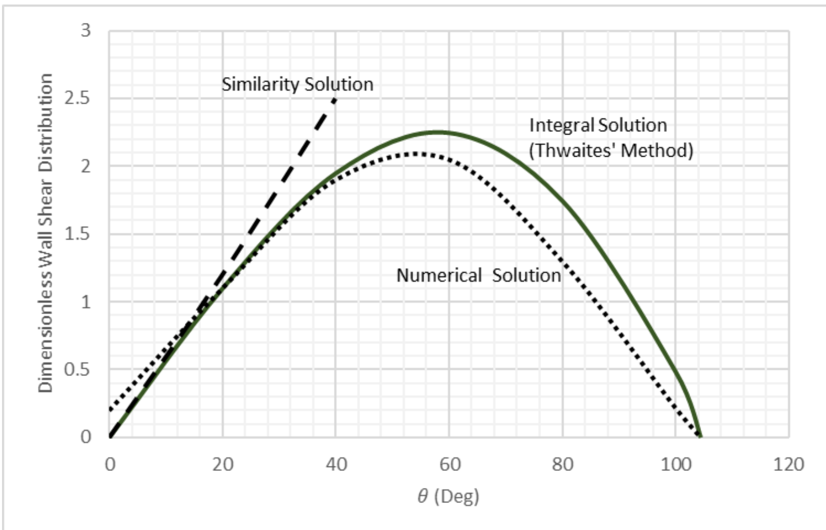

Fig. 9. Variation of shear stress (τ) versus θ obtained from the numerical solution, Thwaites' method and integral solution.

The results of the numerical solution demonstrate good consistency with the integral solution. Also, the numerical solution result is in agreement with the similarity solution from $\theta = 0^{\circ}$ to near stagnation point ($\theta = 30^{\circ}$).

7. Conclusions

In this study, a numerical solution is investigated on a non-similar two-dimensional boundary-layer flow over a cylinder. The solution of the numerical method using the fourth-order Runge-Kutta method and Falkner-Skan equation with the use of MATLAB Software gives accurate outputs. Results were shown in diagrams including friction coefficient (c_f) versus θ angle, and shear stress versus θ angle and x. The results of numerical solution demonstrate good consistency with analytic solutions.

Nomenclature

Re	Reynolds number
Re_{cr}	Critical Reynolds number
r	Cylindrical coordinate
x	Streamwise coordinate
u	Streamwise velocity component
u_e	External flow velocity
u_∞	Cross flow velocity
y	Cartesian coordinate
Ψ	Stream function
θ	Angle between u_∞ and r_0 directions
η	Similarity variable
λ	Thwaites' pressure-gradient parameter
ν	Transverse velocity component
δ	Boundary layer thickness
τ_w	Shear stress at wall
BDF	Backward Differentiation Formula
sep	Subscript for Separation Point

Appendix I:

MATLAB code for Figs. 3, 4, 5 and 6:

```
clc
clear
close all
%% Parameter Spcification
boundary_eta=8;
boundry_x=4.1;
boundary_teta=105;
L = 10;
h = 0.01;
eta = 0:h:L;
teta=linspace(0,105*pi/180,numel(eta));
Inf_init_val=1;
y1(1) = 0;
```

```
y2(1) = 0;
y2(numel(eta))=Inf_init_val;
y3_guess=-2:0.001:2;

mm=[-.2];
yy=.1;
Beta_vec=(2.*mm)./(mm+1);
nu=15.7e-6;
for index=1:numel(Beta_vec)
index
beta = Beta_vec(index)-.01;
m=mm(index)-.01;
x=(eta.*nu./(yy)).^(-(m-1)/2);
r=x./teta;
Rx=x.^(m+1)./nu;

%% Falkner Skan Solution with Runge-Kutta Method
f1 = @(ehta, y1, y2, y3) y2;
f2 = @(ehta, y1, y2, y3) y3;
f3 = @(source,ehta, y1, y2, y3) (source)-((m+1)/2)*y1*y3-m*(1-y2^2);
clear('Y1','Y2','Y3')
for guess=1:numel(y3_guess)
    guess
    y3(1)=y3_guess(guess);

for i = 1:(length(eta)-1)




    a = h.*[f1(eta(i), y1(i), y2(i), y3(i)), f2(eta(i), y1(i), y2(i), y3(i)),
f3(fcn_added(x,i,y1,y2,y3),eta(i), y1(i), y2(i), y3(i))];
    b = h.*[f1(eta(i), y1(i)+a(1)/2, y2(i)+a(2)/2, y3(i)+a(3)/2), f2(eta(i)+h/2,
y1(i)+a(1)/2, y2(i)+a(2)/2, y3(i)+a(3)/2), f3(fcn_added(x,i,y1,y2,y3),eta(i)+h/2,
y1(i)+a(1)/2, y2(i)+a(2)/2, y3(i)+a(3)/2)];
    c = h.*[f1(eta(i), y1(i)+b(1)/2, y2(i)+b(2)/2, y3(i)+b(3)/2), f2(eta(i)+h/2,
y1(i)+b(1)/2, y2(i)+b(2)/2, y3(i)+b(3)/2), f3(fcn_added(x,i,y1,y2,y3),eta(i)+h/2,
y1(i)+b(1)/2, y2(i)+b(2)/2, y3(i)+b(3)/2)];
    d = h.*[f1(eta(i), y1(i)+c(1), y2(i)+c(2), y3(i)+c(3)), f2(eta(i)+h, y1(i)+c(1),
y2(i)+c(2), y3(i)+c(3)), f3(fcn_added(x,i,y1,y2,y3),eta(i)+h, y1(i)+c(1), y2(i)+c(2),
y3(i)+c(3))];
    y3(i+1) = y3(i)+ 1/6*(a(3)+2*b(3)+2*c(3)+d(3));
    y2(i+1) = y2(i)+ 1/6*(a(2)+2*b(2)+2*c(2)+d(2));
    y1(i+1) = y1(i)+ 1/6*(a(1)+2*b(1)+2*c(1)+d(1));
end

Y1(:,guess)=y1;
Y2(:,guess)=y2;
Y3(:,guess)=y3;
end
[value, idx_guess] = min(abs(Inf_init_val-Y2(end,:)));
value
idx_guess
y1=Y1(:,idx_guess);
y2=Y2(:,idx_guess);
y3=Y3(:,idx_guess);
f_double(index)=y3(1);
x=x';
taw=1*(x.^(m+.3)).*y3.*sqrt(x.^(m+.3)/nu);
taw=2.1*taw./max(taw);
Rx=Rx';
cf=2*f_double(index)./sqrt(Rx);
%cf(find(cf==Inf))=1;
```

```
cf(1)=cf(2);
cf=cf./max(cf);
%% Plotting
Legendinfo{index} = sprintf('m=%.5f',mm(index));

figure(3)
plot(eta,y1,'LineWidth', 2)
xlabel('\eta', 'FontSize', 20);
ylabel('f', 'FontSize', 20);
grid on
hold on;

figure(4)
plot(eta,y2,'LineWidth', 2)
xlabel('\eta', 'FontSize', 20);
ylabel('df','FontSize', 20);
grid on
hold on;

figure(5)
plot(eta,y3,'LineWidth', 2)
xlabel('\eta', 'FontSize', 20);
ylabel(' d^{2} f/d\eta^{2}' ,'FontSize', 20);
grid on
hold on;

figure(6)
plot(teta*105/50*180/pi,cf,'LineWidth', 2)
xlabel('\theta', 'FontSize', 20);
ylabel('Normalized Cf', 'FontSize', 20);
% '(u_{\infty}r_{0}/\nu}/pu_{\infty^{2} } [deg]'
grid on
hold on;

figure(7)
plot(x*150*105/50,taw,'LineWidth', 2)
xlabel('x', 'FontSize', 20);
ylabel('\tau(u_{\infty}r_{0}/\nu)^{0.5}(\rhou_{\infty}^{2})^{-1}', 'FontSize', 20);
% '(u_{\infty}r_{0}/\nu}/pu_{\infty^{2} } [deg]'
grid on
hold on;

figure(8)
plot(teta*105/50*180/pi,taw,'LineWidth', 2)
xlabel('\theta', 'FontSize', 20);
ylabel('\tau(u_{\infty}r_{0}/\nu)^{0.5}(\rhou_{\infty}^{2})^{-1}', 'FontSize', 20);
% '(u_{\infty}r_{0}/\nu}/pu_{\infty^{2} } [deg]'
grid on
hold on;


end

figure(3)
legend(Legendinfo, 'FontSize', 15)
xlim([0 boundary_eta])

figure(4)
%legend(Legendinfo, 'FontSize', 15)
xlim([0 boundary_eta])

figure(5)
legend(Legendinfo, 'FontSize', 15)
```

```
xlim([0 boundary_eta])

figure(6)
%legend(Legendinfo, 'FontSize', 15)

figure(7)
%legend(Legendinfo, 'FontSize', 15)
xlim([0 boundry_x])

figure(8)
%legend(Legendinfo, 'FontSize', 15)
xlim([0 boundary_teta])


function[source]= fcn_added(x,i,y1,y2,y3)
h = 0.01;
if i==1
source(i)=x(i)*((y2(i)*(y2(i)-0))-(y3(i)*(y1(i)-0)))/h;
else
source(i)=x(i)*((y2(i)*(y2(i)-y2(i-1)))-(y3(i)*(y1(i)-y1(i-1))))/h;
end

end
```

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