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ASYMPTOTIC SOLUTION OF FOKKER-PLANCK EQUATION BASED ON DARCY-BUCKINGHAM APPROACH IN UNSATURATED SOIL

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Abstract

The present study discusses one dimensional groundwater infiltration phenomenon in unsaturated soil. The mathematical formulation is developed using Dracy's law, equation of continuity, and standard relations between diffusivity coefficient and hydraulic conductivity. We solved the obtained system using double power series method in the form of asymptotic expansion which represents moisture content in soil.

Keywords: Fokker-Planck equation, asymptotic solution, groundwater, infiltration, double power series method.

1. Introduction

Groundwater is the term used to denote all water found beneath the ground surface. Groundwater recharge is the process of water infiltrating from surface and percolating to water table. Generally, groundwater is recharged naturally by rain, snow melt, and, to a smaller extent, by surface water (lakes and rivers).

Moisture/Water content is a finite amount of water stored in a porous material (soil, fruit, rock, ceramics, or wood). The major amount of moisture is found in pores of such materials. The preferred areas where water could be stored are within pore spaces in between and within porous material aggregate spaces. The classification of unsaturated, partially saturated, and saturated porous materials explicitly depends on the quantity of stored air or water in pores. The amount of moisture (water in liquid and/or vapor phase) in a soil is often described in terms of volumetric moisture content. The volumetric moisture content is merely the ratio of water volume to porous material volume.

The mathematical model for moisture content is developed in the form of nonlinear partial differential equation which is known as Fokker-Planck diffusion-convection equation commonly used to describe one dimensional, nonhysteretic infiltration in uniform nonswelling soil.

The study of one dimensional groundwater infiltration phenomenon is of great importance for soil science, hydrologists, agriculturists, civil engineering etc. This problem has been discussed by many researchers with different viewpoints and various methods have been used such as Homotopy Perturbation Summudu Transform Method (HPSTM) (Choksi and Singh 2017), Optimal Homotopy Analysis Method (OHAM) (Pathak and Singh 2016), Homotopy Analysis Method (HAM) (Patel et al. 2014), Similarity Transformation (Parikh et al. 2012, Patel et al. 2012) to obtain numerical and approximate solutions.

Furthermore, Gupta and Kanth gave comparative study of New Homotopy Pertibution Method (NHPM) and Finite Difference Method (FDM) for solving unsteady heat conduction equation (Gupta and Kanth, 2021) and Mingling applied Renormalization Group Method (RGM) for solving a class of Lagrange mechanical system (Mingling, 2022).

The main aim of this paper is to use the Double Power Series Method (DPSM) (Makwana and Parikh, 2017) to achieve an approximation solution of one dimensional nonlinear Fokker-Planck equation based on Darcy Buckingham approach. Using DPSM, we get an infinite series solution. Other authors have studied DPSM (Nuseir and Al-Hasoon 2012; Roopashree and Nargund 2016) to solve various kinds of NPDEs.

The present paper consists of four sections; it starts with an introduction wherein the past studies in the field are reviewed. In Section 2, the problem has been described mathematically. The implementation of DPSM, the validation results, graphical and numerical presentations are all included in Section 3. Finally, Section 4 present the overall conclusions.

2. Mathematical formulation of the problem

When water moves through unsaturated soil, the volume of flow of water as per Darcy's law (Darcy 1856) is:

$$\vec{V} = -K(\theta)\nabla H \tag{1}$$

For continuous flow of water in unsaturated soil, the mass conservation law is given by:

$$(\rho_b \phi S)_t = -\nabla M \tag{2}$$

Using the incompressibility of water, $M = \rho \vec{V}$ and keeping in mind the moisture content $\theta = \phi S$ (Bear 1972), equation (2) reduces to

$$(\rho_b \theta)_t = -\nabla(\rho \vec{V}) \tag{3}$$

From equation (1) and equation (3), we obtain:

$$(\rho_b \theta)_t = -\nabla[\rho\{-K(\theta)\,\nabla H\}] \tag{4}$$

The flow takes place only in one dimension (Scheidegger, 1957), so equation (4) can be rewritten as:

$$\rho_b\ \theta_t = \rho\,[(K(\theta)\cdot H_z)_z] \tag{5}$$

For unsaturated soil, the whole soil moisture potential is given by $H = \psi - gz$ (Scheidegger 1957). Hence, equation (5) can be written as:

$$\theta_t = \frac{\rho}{\rho_b}[(K(\theta)\cdot\psi_z)_z] - \frac{\rho\, g}{\rho_b}(K(\theta))_z \tag{6}$$

By means of chain rule,

$$\psi_z = \psi_\theta \cdot \theta_z \text{ and } (K(\theta))_z = K_\theta \cdot \theta_z \tag{7}$$

If K and ψ are single-valued functions of θ, then

$$D(\theta) = \frac{\rho}{\rho_b}\cdot K(\theta)\cdot \psi_\theta \tag{8}$$

Substituting equations (7) and (8) in to equation (6), we obtain

$$\theta_t = (D(\theta)\cdot\theta_z)_z - \frac{\rho\, g}{\rho_b}\cdot K_\theta \cdot \theta_z \tag{9}$$

Which is known as one dimensional nonlinear Fokker-Planck equation based on Darcy-Buckingham approach.

In view of convenience, here we consider that the infiltration takes place only one dimensional from the top of the surface $z = 0$ to the bottom $z = L$.

To convert equation (9) in dimensionless form, we choose

$$Z = \frac{z}{L}, T = \frac{\rho g}{\rho_b L} t$$

On account of this, we obtain a dimensionless form of equation (9) as:

$$\theta_T = \varepsilon\,[(D(\theta)\cdot\theta_Z)_Z] - K_\theta \cdot \theta_Z \tag{10}$$

where $\varepsilon = \dfrac{\rho_b}{\rho g L}$.

According to Broadbridg and White (1988) model, soil water diffusivity and hydraulic conductivity are given by:

$$D(\theta) = a\,(b-\theta)^{-2} = a\,b^{-2}\,(1-\theta b^{-1})^{-2} \tag{11}$$

$$K(\theta) = \beta + \gamma\,(b-\theta) + \frac{\lambda}{2}(b-\theta)^{-1} \tag{12}$$

where a, b, β, γ and λ are constant.

Since θ is very small, the equations (11) and (12) can be expressed as:

$$D(\theta) = a\,b^{-2}\,(1+2\theta b^{-1}) \tag{13}$$

$$K(\theta) = \beta + \gamma b + \frac{\lambda}{2}b^{-1} + \left(\frac{\lambda}{2}b^{-2} - \gamma\right)\theta \tag{14}$$

Using equations (13) and (14) in to (10), we obtain:

$$\theta_T = \varepsilon\, a\, b^{-3}\, [\{(b+2\theta)\, \theta_Z\}_Z] - \left(\frac{\lambda}{2} b^{-2} - \gamma\right)\theta_Z \tag{15}$$

The equation (15) is a governing equation of the moisture content for unsaturated soil. Equation (15) can be rewritten as

$$\theta_T = A\,[(\theta_Z)^2 + \theta\cdot\theta_{ZZ}] + B\,\theta_{ZZ} - C\,\theta_Z \tag{16}$$

where $A = 2\varepsilon ab^{-3}, B = \varepsilon ab^{-2}$ and $C = \left(\frac{\lambda}{2} b^{-2} - \gamma\right)$.

Consider the appropriate initial and boundary conditions

$$\theta(Z,0) = \theta_0 e^Z,\quad 0 < T \tag{17}$$

$$\theta(0,T) = \theta_0,\quad 0 < Z \tag{18}$$

3. Implementation of double power series method

Let double power series solution of equation (16) be of the form

$$\theta(Z,T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\, Z^m\, T^n \tag{19}$$

Differentiating equation (19) partially with respect to Z and T, we get double series expansion of θ_Z and θ_T are as follows:

$$\theta_Z = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m+1)\, c_{(m+1)n}\, Z^m\, T^n \tag{20}$$

$$\theta_T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (n+1)\, c_{m(n+1)}\, Z^m\, T^n \tag{21}$$

Also, the double series expansion of $(\theta_Z)^2$, θ_{ZZ} and $\theta\cdot\theta_{ZZ}$ are as follows:

$$(\theta_Z)^2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{p=0}^{n}\sum_{q=0}^{m} (q+1)(m-q+1) c_{(q+1)p} c_{(m-q+1)(n-p)}\right] Z^m\, T^n \tag{22}$$

$$\theta_{ZZ} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m+1)(m+2)\, c_{(m+2)n}\, Z^m\, T^n \tag{23}$$

$$\theta\cdot\theta_{ZZ} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{p=0}^{n}\sum_{q=0}^{m} (m-q+1)(m-q+2) c_{qp} c_{(m-q+2)(n-p)}\right] Z^m\, T^n \tag{24}$$

Replacing the derivatives θ_Z, θ_T, $(\theta_Z)^2$, θ_{ZZ} and $\theta\cdot\theta_{ZZ}$ in equation (16) by corresponding double series from equations (20) to (24) respectively, we get the recurrence relation as:

$$
\begin{aligned}
(n+1)c_{m(n+1)} = A\Big[\sum_{p=0}^{n}\sum_{q=0}^{m}\big\{(q+1)(m-q+1)\, c_{(q+1)p}c_{(m-q+1)(n-p)} + (m-q+1)(m-q+2)\, c_{qp}c_{(m-q+2)(n-p)}\big\}\Big] \\
+B(m+1)(m+2)c_{(m+2)n} - C(m+1)c_{(m+1)n}, \ \forall\, m,n \geq 0
\end{aligned} \tag{25}
$$

The initial condition (17) and boundary condition (18) are transformed as follows:

$$c_{m0} = \frac{1}{m!}\theta_0, \quad \forall \ m \geq 0 \tag{26}$$

$$c_{0n} = \begin{cases} \theta_0, & \text{if } n = 0 \\ 0, & \text{if } n \geq 1 \end{cases} \tag{27}$$

Substituting equations (26) and (27) into equation (25) for several values of m and n, and by recursive method, we obtain the following coefficients:

$$c_{11} = 4A\theta_0^2 + B\theta_0 - C\theta_0,\ c_{21} = \frac{1}{2!}(8A\theta_0^2 + B\theta_0 - C\theta_0),$$

$$c_{31} = \frac{1}{3!}(16A\theta_0^2 + B\theta_0 - C\theta_0),\ c_{41} = \frac{1}{4!}(32A\theta_0^2 + B\theta_0 - C\theta_0),$$

and so on.

Substituting all c_{mn} into equation (19), we obtain the following infinite series:

$$
\begin{aligned}
\theta(Z,T) = \theta_0 + \theta_0 Z + \frac{1}{2!}\theta_0 Z^2 + \frac{1}{3!}\theta_0 Z^3 + \frac{1}{4!}\theta_0 Z^4 + \ldots + (4A\theta_0^2 + B\theta_0 - C\theta_0)ZT \\
+\frac{1}{2!}(8A\theta_0^2 + B\theta_0 - C\theta_0)Z^2T + \frac{1}{3!}(16A\theta_0^2 + B\theta_0 - C\theta_0)Z^3T \\
+\frac{1}{4!}(32A\theta_0^2 + B\theta_0 - C\theta_0)Z^4T + \ldots
\end{aligned} \tag{28}
$$

which is the solution of equation (16). The numerical solution of equation (28) is shown in Table 1 and its graphical representation are shown in Fig. 1 and Fig. 2. Table 2 and Fig. 3 indicate the comparison between the various results acquired by various methods such as DPSM, OHAM and HAM.

According to a convergence of double power series given by Ghorpade and Limaye (2010), substitute $T = 0$ in equation (28), we get:

$$\theta(Z,0) = \theta_0 + \theta_0 Z + \frac{1}{2!}\theta_0 Z^2 + \frac{1}{3!}\theta_0 Z^3 + \frac{1}{4!}\theta_0 Z^4 + \ldots$$

which is convergent. Hence, we say that the double power series obtained on right hand side of (28) is also convergent.

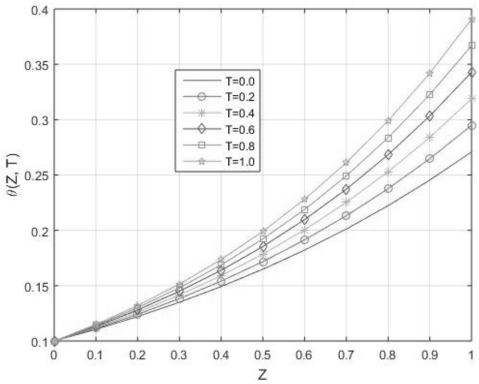

Fig. 1. 2D plot of Moisture Content $\theta(Z,T)$ Vs. Depth (Z).

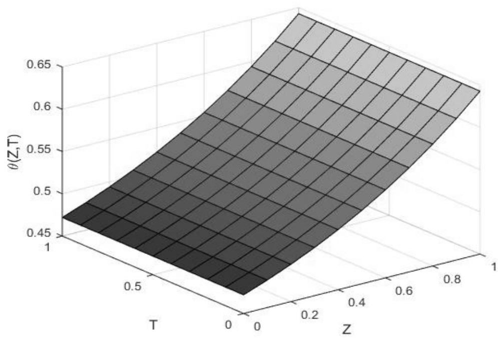

Fig. 2. 3D plot of Moisture Content $\theta(Z,T)$ Vs. Depth (Z).

Z	Moisture Content $\theta(Z,T)$					
	$T=0.0$	$T=0.2$	$T=0.4$	$T=0.6$	$T=0.8$	$T=1.0$
0.0	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
0.1	0.1105	0.1114	0.1123	0.1132	0.1141	0.1149
0.2	0.1221	0.1214	0.1261	0.1280	0.1300	0.1320
0.3	0.1350	0.1318	0.1416	0.1448	0.1481	0.1514
0.4	0.1492	0.1541	0.1590	0.1638	0.1687	0.1736
0.5	0.1648	0.1717	0.1785	0.1853	0.1922	0.1990
0.6	0.1821	0.1913	0.2005	0.2097	0.2189	0.2280
0.7	0.2012	0.2132	0.2252	0.2372	0.2492	0.2612
0.8	0.2222	0.2376	0.2529	0.2683	0.2836	0.2990
0.9	0.2454	0.2647	0.2840	0.3033	0.3227	0.3420
1.0	0.2708	0.2948	0.3188	0.3428	0.3668	0.3908

Table 1. Numerical values of the moisture content θ to depth (Z) and time (T), taking $A=B=C=1\ \&\ \theta_0=0.1$ are fixed.

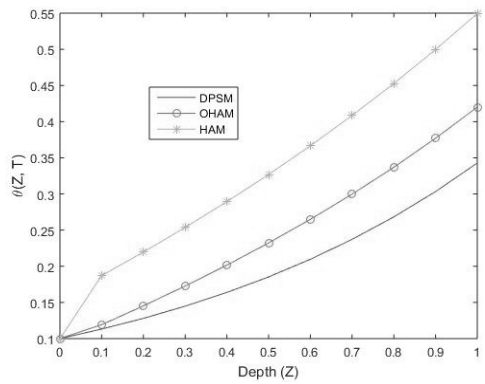

Fig. 3. Comparison of solutions of moisture content obtained by DPSM, OHAM and HAM for fixed time $T = 0.6$.

Z	$T = 0.6$		
	DPSM	OHAM (Pathak and Singh 2016)	HAM (Patel et al. 2014)
0.0	0.1000	0.1000	0.1000
0.1	0.1132	0.1192	0.1871
0.2	0.1280	0.1453	0.2196
0.3	0.1448	0.1727	0.2537
0.4	0.1638	0.2016	0.2894
0.5	0.1853	0.2323	0.3269
0.6	0.2097	0.2649	0.3666
0.7	0.2372	0.2997	0.4085
0.8	0.2683	0.3369	0.4527
0.9	0.3033	0.3770	0.4997
1.0	0.3428	0.4201	0.5496

Table 2. Comparison of numerical values of the moisture content obtained by DPSM, OHAM and HAM for fixed time $T = 0.6$.

4. Conclusions

Here, we have studied groundwater infiltration phenomenon in unsaturated homogeneous porous media in vertical downward direction. The solution of nonlinear Fokker Planck diffusion-convection model was provided using DPSM. The obtained results are compared with well-known methods OHAM and HAM. This can be easily verified from Table 2 and Fig. 3. It can be concluded that DPSM functions well for solving any nonlinear phenomenon arising in fluid flow through porous media.

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Nomenclature

$\vec{V}$	Volume flux of moisture
$K(\theta)$	Coefficient of the volumetric water content
∇H	Gradient of the total moisture potential
ρ_b	Bulk density of the medium
ϕ	Porosity
S	Saturation of water
M	Mass of flux of the water
ρ	Flux density
ψ	Pressure potential
z	Evaluation in the vertical downward direction of flow
g	Gravitational constant
$D(\theta)$	Diffusivity coefficient of the moisture content

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