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RENORMALIZATION GROUP METHOD FOR A CLASS OF LAGRANGE MECHANICAL SYSTEMS

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Abstract

Considering the important role of small parameter perturbation term in mechanical systems, the perturbed dynamic differential equations of Lagrange systems are established. The basic idea and method of solving ordinary differential equations by normal renormalization group method are transplanted into a kind of Lagrange mechanical systems, the renormalization group equations of Euler-Lagrange equations are obtained, and the first-order uniformly valid asymptotic approximate solution of Lagrange systems with a single-degree-of-freedom is given. Two examples are used to show the calculation steps of renormalization group method in detail as well as to verify the correctness of the method. The innovative finding of this paper is that for integrable Lagrange systems, its renormalization group equations are also integrable and satisfy the Hamilton system's structure.

Keywords: Renormalization group method, Lagrange system, uniformly valid asymptotic expansion, integrable

1. Introduction

As we all know, the Lagrange mechanical system is a very important ordinary differential equations (Li et al. 2006), which have important applications in many fields, especially in engineering mechanics and physics. Some parameters in many practical mechanical systems often change slightly with the change of displacement, velocity and time. That is to say, mechanical systems contain small parameters (Lou et al. 2016). These systems are closer to actual mechanical systems, for example, the dynamic equations of hybrid robot can be decomposed into rigid motion with slow change and elastic vibration with fast change by perturbation (Zhang et al. 2016). Therefore, the integral theory and response characteristic analysis of Lagrange mechanical systems with perturbation nonlinearity is a very important subject. The perturbation theory is formed in the process of mechanics. It mainly uses the asymptotic expansion containing small parameter to approximate the global quantitative analysis, but the secular terms often appear in the perturbation algorithm, which results in the approximate solution not being uniformly valid. Many scholars have developed many singular perturbation methods such as average method, multi-scale method, matching asymptotic expansion method, WKB asymptotic method and central manifold principle according to Hu (2010), Hinch (1991), Carr (1981, 1983), Golden (1989), Murdock (1991), Lin (1995). While these traditional singular perturbation methods have their own limitations, for example, the multi-scale method is not easy to determine the time scale, the location and thickness of boundary layer of the matching asymptotic expansion method are not easy to determine, and appearing fractional idempotent of small parameter.

In recent years, many research results show that the renormalization group theory is simpler and more effective than the traditional method in dealing with singular perturbation problems according to Chen (1996) and Ziane (2000). Firstly, it does not need to make special assumptions about the structure of perturbation sequence and does not use asymptotic matching when constructing asymptotic expansion. It generates its own asymptotic sequence for the problem. Secondly, it can automatically introduce appropriate evaluation functions, which can avoid the appearance of the fractional index or logarithmic index in traditional methods. At present, the renormalization group method for solving nonlinear ordinary and partial differential equations has achieved many results, such as those described in Kunihir (1998), Shin-ichiro (2000), Kai (2016), Liu (2017) and Wu (2014) but its application in classical mechanical systems is still not prominent. Generally speaking, the model of mechanical system has more practical engineering background. Mathematically, the law of motion of mechanical system is usually expressed in the form of differential equations. Therefore, it is significant to solve the non-linear problems in mechanical systems by renormalization group method.

In this paper, the renormalization group (RG) method is applied to a classical Lagrange mechanical system, and the structure of RG equation and the properties of perturbation solution are analyzed. Specifically, we firstly established the Lagrange equation of non-conservative mechanical system, and secondly, we gave the construction steps of RG algorithm. Finally, we showed the calculation results with two examples which suggesed that RG method can be used to solve uniformly valid asymptotic expansion solution of Lagrange mechanical system.

2. Perturbation model of Lagrange mechanical system

If there are some small parameters which cannot be ignored in Lagrange system, such as time constant, inertia, conductance or capacitance, which can reflect certain physical properties (Zhang, 2013), it can be regarded as a perturbed differential system. Assuming that the configuration of system is determined by n generalized coordinates $q_s(s=1,...,n)$, the Lagrange function is $L(t,\mathbf{q},\dot{\mathbf{q}},\varepsilon)$, so we study a class of perturbed Lagrange systems with the non-conservative forces:

$$L = T - V, Q_s(t,\mathbf{q},\dot{\mathbf{q}},\varepsilon), (s=1,2,...,n). \tag{1}$$

Here, kinetic energy T, potential energy V and non-conservative force Q_s are all analytic functions, $0<\varepsilon<<1$ is a small parameter.

The differential equation of motion can be expressed as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s. \tag{2}$$

Assuming that the system is not singular that is $\det\left[\frac{\partial^2 T}{\partial \dot{q}_s \partial \dot{q}_k}\right] \neq 0$, so the all generalized accelerations are:

$$\ddot{q}_s = \alpha_s(t, q_s, \dot{q}_s, \varepsilon). \tag{3}$$

Further, we introduce the generalized momentum $p_s = \frac{\partial L}{\partial \dot{q}_s}$, so the Hamiltonian function of equation (1) is $H(q_s, p_s, \varepsilon) = \sum_{i=1}^{n} \dot{q}_i p_i - L$, so for the equation (2), the Hamilton's canonical equations with the non-conservative forces are：

$$\begin{cases} \dot{q}_s = \frac{\partial H}{\partial p_s}, \\ \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s, \end{cases} (s = 1, 2, .., n). \tag{4}$$

The equations (4) represent the system's Hamilton structure.

3. Renormalization group theory

The basic idea of the renormalization group method is to first construct the direct perturbation expansion of the nonlinear equation, and then eliminate the secular term in the direct perturbation expansion by introducing free parameters, so as to obtain a uniformly effective asymptotic expansion. Now, we introduce the concrete steps of renormalization group theory which are applied to mechanical systems using the renormalization group method of non-linear differential equation. Considering a general form of perturbation differential equation:

$$L(y) = \varepsilon f(t, y, y', ...), \tag{5}$$

Here, L is a linear or non-linear differential operator. We expand y by the perturbation:

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + ..., \tag{6}$$

Here, $y_0, y_1, y_2, ...$ are the functions to be solved. Take equation (6) back to equation (5), then comparing the homogeneous coefficient of ε at both ends of the equation:

$$L(y_0) = 0, L(y_1) = f(t, y_0, y_0', ...), L(y_k) = f(t, y_{k-1}, y_{k-1}', ...), ..., \tag{7}$$

We assume that L is a two-order derivative operator, y_0 contains two undetermined integrals R_0, θ_0, which are not constants in the nonlinear perturbation sequence.

Setting the initial time is t_0, we can derive the one-order asymptotic expansion of solution by perturbation:

$$y(t, t_0) = Y_0(R_0, \theta_0, t) + \varepsilon Y_1(R_0, \theta_0, t, t_0) + O(\varepsilon^2), \tag{8}$$

Considering the secular terms such as $(t - t_0)(\cos t, \sin t)$ in Y_1, in order to keep the uniformly effective approximation, a free parameter τ is introduced, $t - t_0$ is $t - \tau + \tau - t_0$, RG introduces the renormalization of multiplication and addition:

$$R_0(t_0) = Z_1(t_0, \tau)R(\tau), \theta_0(t_0) = \theta(\tau) + Z_2(t_0, \tau), \tag{9}$$

Here, $Z_1 = 1 + \sum_1^\infty a_n \varepsilon^n, Z_2 = \sum_1^\infty b_n \varepsilon^n$, a_n, b_n are undetermined constants.

Substituting the equation (9) to the equation (8), the approximate steady-state solution

$y(R,\theta,,t,\tau,\varepsilon)$ of $y(t)$ is obtained by selecting the renormalization constants a_n, b_n appropriately and eliminating the $\tau - t_0$ contained in expansion, because $y(t)$ does not depend on τ , so the renormalization equation is:

$$\frac{\partial y(R,\theta,,t,\tau,\varepsilon)}{\partial \tau} = 0. \tag{10}$$

Noticing that the further expression of equation (10) is:

$$\begin{aligned} \frac{dR}{d\tau} &= g(\tau, R, \theta, \varepsilon) + O(\varepsilon^2), \\ \frac{d\theta}{d\tau} &= h(\tau, R, \theta, \varepsilon) + O(\varepsilon^2), \end{aligned} \tag{11}$$

Solving $R(\tau), \theta(\tau)$, and changing τ into t , the bounded uniformly valid asymptotic expansion is:

$$y(t) = y(R(t), \theta(t), t, \varepsilon, y(t_0), y'(t_0)). \tag{12}$$

In principle, for the higher-order uniformly asymptotic expansions, the divergence will occur at the each order of ε . In this way, we can choose the renormalization constants to cancel the divergence term at each-order and discuss the structure of renormalization group. We can expect that if the infinite order can be achieved, it will be consistent with the exact solution of original equation.

4. Examples illustration

Example 1 The Lagrange function and non-conservative force of a single-degree-of-freedom perturbation dynamic system are:

$$L(t,q,\dot{q},\varepsilon) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 - \frac{1}{2}\varepsilon q^2, \ \ Q = 0, \tag{13}$$

Solving the uniform asymptotic expansion solution by using renormalization group method.

According to the Lagrange equation (2) of mechanical system, we can obtain:

$$\ddot{q} = -q - \varepsilon q. \tag{14}$$

Its direct perturbation expansion is:

$$q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \ldots, \tag{15}$$

By substituting equation (15) into equation (14), the homogeneous coefficient of ε at both ends of the equation is equal, so:

$$\varepsilon^0 : \frac{d^2 q_0}{dt^2} + q_0 = 0,$$
$$\varepsilon^1 : \frac{d^2 q_1}{dt^2} + q_1 = -q_0, \tag{16}$$
$$\varepsilon^2 : \frac{d^2 q_k}{dt^2} + q_k = -q_{k-1}, k = 2,3,...,$$

So, the solution is:

$$q_0 = B\cos t + C\sin t,$$
$$q_1 = \bar{B}\cos t + \bar{C}\sin t + \frac{(t-t_0)}{2}(C\cos t - B\sin t), \tag{17}$$

Here, $B, C, \bar{B}, \bar{C}$ are arbitrary constants.

Without loss of generality, set $\bar{B} = 0, \bar{C} = 0$, so:

$$q(t) = B\cos t + C\sin t + \frac{\varepsilon}{2}(t-t_0)(C\cos t - B\sin t) + O(\varepsilon^2), \tag{18}$$

In order to eliminate the secular terms, introducing the any moment μ , the renormalization quantity is:

$$B(t_0) = (1 + b_1\varepsilon)B(\mu), C(t_0) = (1 + c_1\varepsilon)C(\mu), \tag{19}$$

By substituting equation (19) into equation (18), in order to eliminate the μ-t_0 contained in the exhibition, the coefficients are selected as follows:

$$b_1 = -\frac{C}{2B}(\mu - t_0), c_1 = \frac{B}{2C}(\mu - t_0). \tag{20}$$

Substitution equation (18), we can get:

$$q(t) = B(\mu)\cos t + C(\mu)\sin t + \frac{\varepsilon}{2}(t-\mu)(C(\mu)\cos t - B(\mu)\sin t) + O(\varepsilon^2), \tag{21}$$

The renormalization group equation of equation (14) is determined by $\frac{\partial q}{\partial \mu} = 0$, so the renormalization group equation is:

$$\frac{dB}{d\mu} = \frac{\varepsilon}{2}C(\mu), \frac{dC}{d\mu} = -\frac{\varepsilon}{2}B(\mu). \tag{21}$$

An important discovery, here, it should be emphasized that if the B, C are regarded as generalized coordinate and momentum, the Hamiltonian function is $H = \frac{\varepsilon}{2}(B^2 + C^2)$, so it is easy to find that the renormalization group equation (22) is a Hamiltonian system with Hamilton structure, further, the original Lagrange system (13) is integrable.

Solving equation (22), considering the initial conditions $q(0), \dot{q}(0)$, set $\mu = t$, the uniformly valid expansion is:

$$q(t) = B(0)\cos(t + \frac{\varepsilon}{2}t) + C(0)\sin(t + \frac{\varepsilon}{2}t) + O(\varepsilon^2), \tag{23}$$

The exact solution of equation (14) is:

$$q = B_0 \cos(\sqrt{1+\varepsilon t}) + C_0 \sin(\sqrt{1+\varepsilon t}). \tag{24}$$

Obviously, the equation (23) is the one-order uniformly efficient approximation of equation (24).

In order to verify the accuracy of RG method, we see the relationship between RG method and exact solution by drawing. Setting $q(0) = 1, \dot{q}(0) = 1$, Fig. 1 shows the error comparison between RG method and numerical solution approximate the exact solution when $\varepsilon = 0.02$. Fig. 2 shows the error comparison between RG method and numerical solution approximate the exact solution when $\varepsilon = 0.002$.

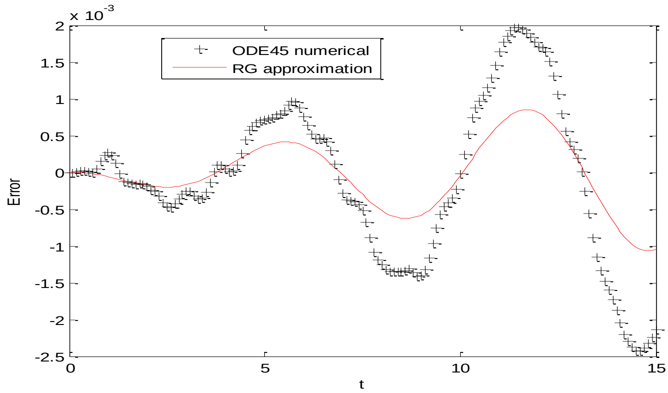

Fig. 1. The comparison between RG method and numerical method when $\varepsilon = 0.02$.

It can be seen from Fig. 1 that, although the value of small parameter ε is large, the error fluctuation of RG method is lower than that of numerical method, therefore, RG method is stable.

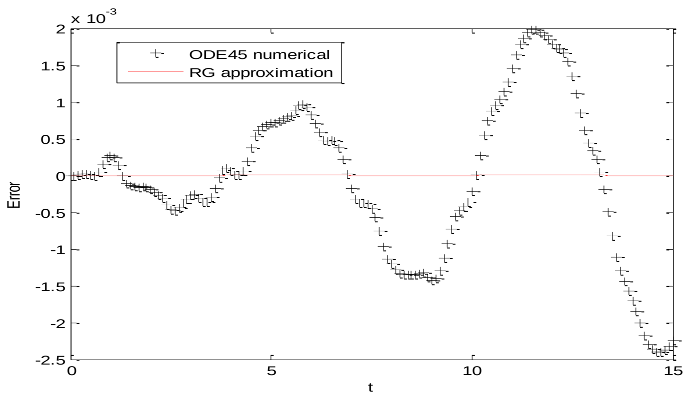

Fig. 2. The comparison between RG method and numerical method when $\varepsilon = 0.002$.

It can be seen from Fig. 2 that, when the small parameter ε is very small, the error of RG method is very close to zero; however, the error of numerical method is still large, therefore, for small parameter perturbed system, the accuracy of RG method is significantly higher than that of numerical algorithm.

Example 2 The Lagrange function and non-conservative force of a single-degree-of-freedom perturbation dynamic system are:

$$L(t,q,\dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2, Q = -\varepsilon\dot{q}, \tag{25}$$

Solving the uniform asymptotic expansion solution by using renormalization group method.

According to the Lagrange equation (2) of non-conservative mechanical system, we can obtain:

$$\ddot{q} = -q - \varepsilon\dot{q}. \tag{26}$$

Repeating the calculation steps in Example 1, the outcome of the first order RG method is:

$$q(t) = B(0)e^{-\frac{\varepsilon}{2}t}(\sin(t + C(0)) - \frac{\varepsilon}{4}\cos(t + C(0))), \tag{27}$$

While the outcome of the second order RG method is:

$$q(t) = B(0)e^{-(\frac{\varepsilon}{2}+\frac{\varepsilon^2}{16})t}(\sin(t - \frac{\varepsilon^2}{8}t + C(0)) - \frac{\varepsilon}{4}\cos(t - \frac{\varepsilon^2}{8}t + C(0))), \tag{28}$$

Importantly, since the original equation (26) is a constant coefficient equation, the exact solution of equation (26) can be found:

$$q(t) = e^{-\frac{\varepsilon}{2}t}(B_0 \sin(t - \frac{\varepsilon^2}{8}t) + C_0 \cos(t - \frac{\varepsilon^2}{8}t)), \tag{29}$$

In order to verify the accuracy of RG method, we see the relationship between RG method and exact solution by drawing. Setting $\varepsilon = 0.5, q(0) = 1, \dot{q}(0) = 0$, Fig. 3 shows how precise the first and second order RG outcomes approximate the exact solution.

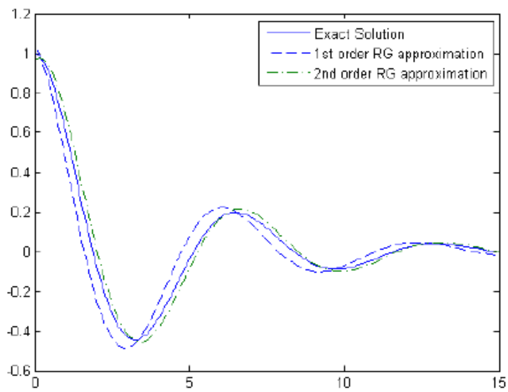

Fig. 3. The relationship between RG method and exact solution.

It can be seen from Fig. 3 that although the value of small parameter ε is large, the RG method is completely consistent with the exact solution curve.

Fig. 4 shows the difference between the first order RG and the exact solution as well as the second order RG and the exact solution.

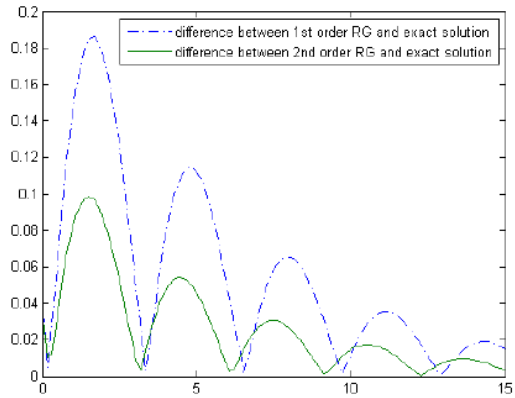

Fig. 4. The error functions for RG method and exact solution.

It can be seen from Fig. 4 that the error of second order RG method is less than the first order RG method, and the error becomes maximum around $t = 3$ and then converges to zero around $t = 15$.

Therefore, we can conclude that the RG method has very good agreement with the exact solution in the large parameter and large time scale range, which shows that the RG method is very powerful in the asymptotic analysis.

5. Conclusions

In this paper, the renormalization group method is used to study the solution of a kind of separable Lagrange perturbation equation in mechanical systems. We obtain the one-order approximate solution, which can be used to simulate analytically law of motion. It is found that the renormalization group algorithm is effective in solving nonlinear differential equations and is easy to program. It can obtain uniformly asymptotic and large-scale global solution. Importantly, the structure of RG equation is closely related to some self-characteristics of mechanical system, such as if the mechanical system is integrable, the RG equation is also integrable and is a Hamilton system. At the same time, two examples are used to illustrate the correctness of RG method. The simulation graphics show that when the parameters is very small, the error of RG method is close to zero, and the accuracy of second-order RG method is higher than that of first-order RG method.

Furthermore, we can use the renormalization group method to study the higher order uniformly valid asymptotic expansion and the influence of parameter sensitivity of perturbed dynamics equation of constrained mechanical system.

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