EXACT AUGMENTED PERPETUAL MANIFOLDS: A COROLLARY FOR THEIR UNIQUENESS

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Abstract

The perpetual points have been defined recently as characteristic points in a dynamical system. In many unexcited linear and nonlinear mechanical systems, the perpetual points are associated with rigid body motions and form the perpetual manifolds. The mechanical systems that admit rigid body motions as solutions are called perpetual. In the externally forced mechanical system, the definition of perpetual points to the exact augmented perpetual manifolds extended. The exact augmented perpetual manifolds are associated with the rigid body motion of mechanical systems but with externally excited. The definition of the exact augmented perpetual manifolds leads to a theorem that defines the conditions of an externally forced mechanical system to be moving as a rigid body. Therefore, it defines the conditions of excitation of only this specific type of similar modes, the rigid body modes. Herein, as a continuation of the theorem, a corollary is written and proved. It mainly states that the exact augmented perpetual manifolds for each mechanical system are not unique and are infinite. In an example of a mechanical system, the theory is applied by considering different excitation forces in two-time intervals. The numerical simulations with the analytical solutions are in excellent agreement, which is certifying the corollary.

Further, due to the different solutions in the two-time intervals, there is a discontinuity in the vector field and the system's overall solution. Therefore, the state space formed by the exact augmented perpetual manifold is nonsmooth. This work is the first step in examining the exact augmented perpetual manifolds of mechanical systems. Further work is needed to understand them, which mathematical space they belong to, considering that nonsmooth functions might form them.

Keywords: Perpetual points, perpetual manifolds, augmented perpetual manifolds, corollary

1. Introduction

The perpetual points (PPs) in mathematics were defined recently in Prasad (2015). The significance of perpetual points in examining the dynamics of systems is ongoing research. The PPs can be determined by considering, in a dynamical system, accelerations, and jerks equal to zero but for nonzero velocities. Currently, there are four, relevant to PPs, directions. The first research direction is about developments of PPs theory with mathematical definitions and experiments to verify them (Brzeski and Virgin 2017, Dudkowski et al 2016, Prasad 2015-2016). In the second research direction, the PPs are used to locate hidden and chaotic attractors

(Chowdhurry 2020, Dudkowski et al 2015-2018, Kuznetsov 2018, Nazaimehr 2017a-b, Yu 2020). The third research direction is relevant to use them for identifying dissipative systems (Brzeski and Virgin 2018, Cang 2017, Jafari et al 2015, Wang et al 2020, Wu et al 2018). The fourth research direction is about using the PPs in mechanics (Georgiades 2020a-b, Georgiades 2021a-c). In unforced mechanical systems, the PPs with rigid body motions are correlated (Georgiades 2020a-b). The mechanical systems admitting rigid body motions are called perpetual mechanical systems (Georgiades 2021a-b). Also, in (Georgiades 2021a-b), the definition of augmented perpetual manifolds is given, leading to a theorem that defines the conditions that a mechanical system, under external forcing, in the exact augmented perpetual manifolds, is moving as a rigid body. In (Georgiades 2021c), a corollary stated and proved that different mechanical systems could have the same exact augmented perpetual manifolds.

Following the proved theorem in Georgiades (2021a-b), herein, a corollary is written and proved. The theory, through an example of a mechanical system with numerical simulations, is certified.

2. Corollary

Corollary - The exact augmented perpetual manifolds are not unique for each discrete perpetual mechanical system, and they are infinite.

Proof:

Considering two different external vectors ($\left\{\boldsymbol{F_i^A}\right\}$, $\left\{\boldsymbol{F_i}^B\right\}$ in a perpetual mechanical system, then the two systems of equations of motion are given by,

$$\left[\boldsymbol{M_{i,j}}\left(t, q_l(t), \dot{q}_m(t)\right)\right]\times\left\{\ddot{\boldsymbol{q}}_{\boldsymbol{i}}(t)\right\}+\left[\boldsymbol{C_{i,j}}\right]\times\left\{\dot{\boldsymbol{q}}_{\boldsymbol{i}}(t)\right\}+\left[\boldsymbol{K_{i,j}}\right]\times\left\{\boldsymbol{q_i}(t)\right\}+ \\ +\left\{\boldsymbol{F_i^{NL}}\left(q_n(t), \dot{q}_o(t)\right)\right\}=\left\{\boldsymbol{F_i^A}\left(t, q_p(t), \dot{q}_q(t)\right)\right\}, \text{ for} \\ m,n,o,p,q\in\{1,2,\ldots,N\},\ i=1,\ldots,N,\ \ j=1,\ldots,N. \tag{1}$$

and,

$$\left[\boldsymbol{M_{i,j}}\left(t, q_l(t), \dot{q}_m(t)\right)\right]\times\left\{\ddot{\boldsymbol{q}}_{\boldsymbol{i}}(t)\right\}+\left[\boldsymbol{C_{i,j}}\right]\times\left\{\dot{\boldsymbol{q}}_{\boldsymbol{i}}(t)\right\}+\left[\boldsymbol{K_{i,j}}\right]\times\left\{\boldsymbol{q_i}(t)\right\}+ \\ +\left\{\boldsymbol{F_i^{NL}}\left(q_n(t), \dot{q}_o(t)\right)\right\}=\left\{\boldsymbol{F_i^B}\left(t, q_p(t), \dot{q}_q(t)\right)\right\}, \text{ for} \\ l,m,n,o,p,q\in\{1,2,\ldots,N\},\ i=1,\ldots,N,\ \ j=1,\ldots,N. \tag{2}$$

whereas,

$\left[\boldsymbol{M_{i,j}}\right]$ is a real $N\times N$ inertia matrix with elements that can be, nonsmooth, nonlinear, time and state depended, functions but having at least one nonzero sum of k-row for all time instants,

$\left[\boldsymbol{K_{i,j}}\right]$ and $\left[\boldsymbol{C_{i,j}}\right]$, are real $N\times N$ constant, stiffness and proportional to velocity vector, matrices,

$\left\{\boldsymbol{F_i^{NL}}\right\}$ is a $N\times 1$ vector of nonlinear internal forces with elements state depended nonlinear functions which can be nonsmooth but single-valued for rigid body motions, and $F_i^{NL}\left(q_s,0\right)=0$ for $q_s\in\mathbb{R}$,

$\left\{\boldsymbol{F_i}\right\}$ is a real $N\times 1$ vector of external forces with elements, time and state dependent, maybe nonlinear and nonsmooth functions.

The system is perpetual which means zero internal forces for rigid body motions, or expressed in mathematical form,

$$\left[\boldsymbol{C_{i,j}}\right]\times\{1\}\cdot\dot{q}_a\left(t\right)+\left[\boldsymbol{K_{i,j}}\right]\times\{1\}\cdot q_a\left(t\right)+\left\{\boldsymbol{F_i^{NL}}\left(q_a\left(t\right),\dot{q}_a\left(t\right)\right)\right\}=\{0\} \tag{3}$$

Since the system is perpetual for rigid body motion equations (1-2) are taking the forms,

$$\begin{gathered}\left[\boldsymbol{M_{i,j}}\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)\right]\times\left\{\ddot{\boldsymbol{q}}_{\boldsymbol{a,A}}\left(t\right)\right\}=\left\{\boldsymbol{F_i^A}\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)\right\}\ \text{for}\\ i=1,\ldots,N,\ \ j=1,\ldots,N,\end{gathered} \tag{4}$$

and,

$$\begin{gathered}\left[\boldsymbol{M_{i,j}}\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)\right]\times\left\{\ddot{\boldsymbol{q}}_{\boldsymbol{a,B}}\left(t\right)\right\}=\left\{\boldsymbol{F_i^B}\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)\right\}\ \text{for}\\ i=1,\ldots,N,\ \ j=1,\ldots,N\end{gathered} \tag{5}$$

Based on the theorem of Georgiades (2021), for a solution in the exact augmented perpetual manifolds, the external forces for each equation should be correlated through,

$$\begin{gathered}F_i^A\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)=\frac{\sum_{j=1}^{N}M_{i,j}\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)\cdot F_k^A\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)}{\sum_{j=1}^{N}M_{k,j}\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)},\\ \text{for}\ \ i,k\in\left\{1,2,\ldots,N\right\},\end{gathered} \tag{6}$$

and for the 2nd type of external forces,

$$\begin{gathered}F_i^B\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)=\frac{\sum_{j=1}^{N}M_{i,j}\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)\cdot F_k^B\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)}{\sum_{j=1}^{N}M_{k,j}\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)},\\ \text{for}\ \ i,k\in\left\{1,2,\ldots,N\right\}.\end{gathered} \tag{7}$$

Replacing equation (6) in equation (4) leads to the differential equation with vector field G_A describing the motion in the exact augmented perpetual manifold for the first type of external forces, and that is given by

$$\begin{gathered}\ddot{q}_{a,A}\left(t\right)=\frac{F_k^A\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)}{\sum_{j=1}^{N}M_{k,j}\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right)}=G_A\left(t,q_{a,A}\left(t\right),\dot{q}_{a,A}\left(t\right)\right),\\ \text{for}\ \ k\in\left\{1,2,\ldots,N\right\}.\end{gathered} \tag{8}$$

Similarly, the replacement of equation (7) in (5) leads to a differential equation with vector field $G_B\left(\neq G_A\right)$ describing the motion in the exact augmented perpetual manifold for the 2nd type of external forces, and that is given by,

$$\ddot{q}_{a,B}\left(t\right)=\frac{F_k^B\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)}{\sum_{j=1}^{N}M_{k,j}\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right)}=G_B\left(t,q_{a,B}\left(t\right),\dot{q}_{a,B}\left(t\right)\right),\quad \text{for } k\in\left\{1,2,\ldots,N\right\}. \tag{9}$$

Therefore, it leads to two different solutions.

In order to make the proof very clear, without losing the generality of the proof, constant inertia matrix is considered with the following two forms of external forces,

$$F_k^A\left(t\right)=\sum_{j=1}^{N_A}A_j\cdot sin\left(\omega_{A,j}\cdot t+\varphi_j\right),\ \text{ with } N_A\in\mathbb{N}, \tag{10}$$

and

$$F_k^B\left(t\right)=\sum_{j=1}^{N_B}B_j\cdot sin\left(\omega_{B,j}\cdot t+\theta_j\right)+\eta\cdot t+c,\ \text{ with } N_B\in\mathbb{N}. \tag{11}$$

The solutions through equations (8), considering the explicit form of the first type of external forces (eq. 10) as follows is obtained,

$$\ddot{q}_{a,A}\left(t\right)=\frac{\sum_{j=1}^{N_A}A_j\cdot sin\left(\omega_j\cdot t+\varphi_j\right)}{\sum_{j=1}^{N}M_{k,j}}, \tag{12}$$

and with direct integration leads to,

$$\begin{aligned}\dot{q}_{a,A}\left(t\right)=&-\sum_{i=1}^{N_A}\frac{A_i}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{A,i}}\cdot\cos\left(\omega_{A,i}\cdot t+\varphi_i\right)\\&+\sum_{i=1}^{N_A}\frac{A_i\cdot\cos\left(\omega_{A,i}\cdot t_0+\varphi_i\right)}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{A,i}}+\dot{q}_{a,A}\left(t_0\right)\end{aligned} \tag{13}$$

$$\begin{aligned}q_{a,A}\left(t\right)=&-\sum_{i=1}^{N_A}\frac{A_i}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{A,i}^2}\cdot\sin\left(\omega_{A,i}\cdot t+\varphi_i\right)+\left(\sum_{i=1}^{N_A}\frac{A_i\cdot\cos\left(\omega_{A,i}\cdot t_0+\varphi_i\right)}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{A,i}}+\right.\\&\left.+\dot{q}_{a,A}\left(t_0\right)\right)\cdot\left(t-t_0\right)+\sum_{i=1}^{N_A}\frac{A_i\cdot\sin\left(\omega_{A,i}\cdot t_0+\varphi_i\right)}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{A,i}^2}+q_{a,A}\left(t_0\right)\end{aligned} \tag{14}$$

The solutions of the 2nd type of external forces through equation (9), considering the explicit form of the external forces (eq. 11) as follows is obtained, as follows,

$$\ddot{q}_{a,B}(t)=\frac{\sum_{j=1}^{N_B}B_j\cdot sin\left(\omega_{B,j}\cdot t+\theta_j\right)+\eta\cdot t+c}{\sum_{j=1}^{N}M_{k,j}}, \tag{15}$$

and with direct integration leads to,

$$\begin{aligned}\dot{q}_{a,B}(t)=&-\sum_{i=1}^{N_B}\frac{B_i}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{B,i}}\cdot\cos\left(\omega_{B,i}\cdot t+\theta_i\right)+\sum_{i=1}^{N_B}\frac{B_i\cdot\cos\left(\omega_{B,i}\cdot t_0+\theta_i\right)}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{B,i}}+\\&+\frac{\eta}{2\cdot\sum_{j=1}^{N}M_{k,j}}\cdot\left(t^2-t_0^2\right)+\frac{c}{\sum_{j=1}^{N}M_{k,j}}\cdot\left(t-t_0\right)+\dot{q}_{a,B}\left(t_0\right)\end{aligned} \tag{16}$$

$$\begin{aligned}q_{a,B}(t)=&-\sum_{i=1}^{N_B}\frac{B_i}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{B,i}^2}\cdot\sin\left(\omega_{B,i}\cdot t+\theta_i\right)+\left(\sum_{i=1}^{N_B}\frac{B_i\cdot\cos\left(\omega_{B,i}\cdot t_0+\theta_i\right)}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{B,i}}+\right.\\&\left.+\dot{q}_{a,B}\left(t_0\right)\right)\cdot\left(t-t_0\right)+\sum_{i=1}^{N_B}\frac{B_i\cdot\sin\left(\omega_{B,i}\cdot t_0+\theta_i\right)}{\sum_{j=1}^{N}M_{k,j}\cdot\omega_{B,i}^2}\\&+\frac{\eta}{6\cdot\sum_{j=1}^{N}M_{k,j}}\cdot\left(t^3-t_0^3\right)+\frac{c}{2\cdot\sum_{j=1}^{N}M_{k,j}}\cdot\left(t^2-t_0^2\right)\\&-\left(\frac{\eta\cdot t_0^2}{2\cdot\sum_{j=1}^{N}M_{k,j}}+\frac{c\cdot t_0}{\sum_{j=1}^{N}M_{k,j}}-\dot{q}_{a,B}\left(t_0\right)\right)\cdot\left(t-t_0\right)+q_{a,B}\left(t_0\right)\end{aligned} \tag{17}$$

Considering any integer number of N_A and the possibility that $N_A\to\infty$ and for $N_B\to\infty$, with $\eta=c=0$, the solutions of the two vector fields are the same and they seem infinite. Also, one more solution for $\eta\neq 0, c\neq 0$ is defined that exists through the 2nd vector field $\left(G_B\right)$, which leads to infinite solutions. Instead of a linear in time function, any other function can be used.

The solutions obtained for both systems of equations, for at least one nonzero amplitude, leads to particle-wave motion. If the constant terms (last two) of velocity in equation (13) are zero, then the motion is a particle-standing wave.

The analysis herein is elementary, and a detailed analysis of the mathematical space of the functions defining the exact augmented perpetual manifolds can be done elsewhere by considering that they might be nonsmooth functions too, which leads to much-complicated analysis.

3. Example of a mechanical system

In this section, an example of a mechanical system is examined. Figure 1 shows the mechanical system, with the following equations of motion,

$$\begin{aligned}&\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \times \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} k_l & -k_l \\ -k_l & k_l \end{bmatrix} \times \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{bmatrix} c_l & -c_l \\ -c_l & c_l \end{bmatrix} \times \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} + \\ &+ \begin{Bmatrix} k_{nl} \cdot (z_1 - z_2)^3 + c_{nl} \cdot tanh\left(b \cdot (\dot{z}_1 - \dot{z}_2)\right) \\ k_{nl} \cdot (z_2 - z_1)^3 + c_{nl} \cdot tanh\left(b \cdot (\dot{z}_2 - \dot{z}_1)\right) \end{Bmatrix} = \begin{Bmatrix} F_1\left(t, z_1, z_2, \dot{z}_1, \dot{z}_2\right) \\ F_2\left(t, z_1, z_2, \dot{z}_1, \dot{z}_2\right) \end{Bmatrix}\end{aligned} \tag{18}$$

It is straightforward to show that the left side of equations (18) corresponds to a perpetual mechanical system. Setting equal displacements and equal velocities, and zero external forcing vector, the accelerations are equal to zero. The equations of jerks, neglecting the external forcing vector, are given by,

$$\begin{aligned}&\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \times \begin{Bmatrix} \dddot{z}_1 \\ \dddot{z}_2 \end{Bmatrix} + \begin{bmatrix} k_l & -k_l \\ -k_l & k_l \end{bmatrix} \times \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} + \begin{bmatrix} c_l & -c_l \\ -c_l & c_l \end{bmatrix} \times \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \\ &+ \begin{Bmatrix} 3 \cdot k_{nl} \cdot (z_1 - z_2)^2 \cdot (\dot{z}_1 - \dot{z}_2) + b \cdot c_{nl} \cdot sech^2\left(b \cdot (\dot{z}_1 - \dot{z}_2)\right) \cdot (\ddot{z}_1 - \ddot{z}_2) \\ 3 \cdot k_{nl} \cdot (z_2 - z_1)^2 \cdot (\dot{z}_2 - \dot{z}_1) + b \cdot c_{nl} \cdot sech^2\left(b \cdot (\dot{z}_2 - \dot{z}_1)\right) \cdot (\ddot{z}_2 - \ddot{z}_1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\end{aligned} \tag{19}$$

Setting accelerations equal to zero, equal velocities, and equal displacements in equation (19) leads that also the jerks are zero. Therefore, the system is forming a perpetual mechanical system.

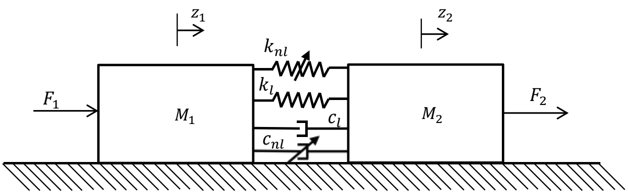

Fig. 1. Configuration of the mechanical system.

The parameters are $m_1 = 2000\,kg$, and $m_2 = 1000\,kg$. The linear coupling stiffness is $k_1 = 10^6\ N/m$ and the nonlinear stiffness is $k_2 = 5 \cdot 10^5\ N/m^3$. Modal analysis of the underlying linear system leads to the first natural frequency is zero, and the 2nd one is $\omega_2 = 38.73\,{rad}/{s}$.

The damping coefficient is $c_1 = 516.398\,N \cdot s/m$, obtained as a 1% damping ratio for the 2nd mode, and the nonlinear damping coefficient has same value $c_2 = 516.398\,N$ with $b = 10^6\ s/m$ (significantly high to approximate dry friction).

Two-time intervals of 1s each are considered. In the first time interval, the external forces are given by equation (10) with $N_A = 100$, and,

$$\omega_{A,i} = i \cdot 2 \cdot \pi\ rad/s, \text{ for } i = 1, \ldots, N_A, \tag{20}$$

$$A_i = i \cdot 10^5\ N, \text{ with } i = 2, \ldots, N_A. \tag{21}$$

In the 2nd time interval, the external forces are given by the equation (11), with $N_B = 150$, and,

$$\omega_{B,i} = \omega_{B,i-1} + 3\ rad/s, \text{ for } i = 2, \ldots, N_B,\ \omega_{B,i-1} = 2 \cdot \pi\ rad/s, \tag{22}$$

$$B_i = B_{i-1} + 1 \cdot 10^6 \ N, \text{ for } i = 2, \ldots, N_B, \ B_1 = 1 \cdot 10^7 \ N. \tag{23}$$

The external forcing vectors are following equations (6-7), and the analytical solution by equations (13-14) and (16-17), are given for the two-time intervals, respectively.

The numerical simulations with Scilab 5.5.2 64-bit (The Scilab Team 2015) using 'Adams' solver with time step $dt = 1 \cdot 10^{-4}$ s, relative and absolute tolerance $1 \cdot 10^{-14}$, have been performed.

The initial conditions are $x(0) = y(0) = 1m$, $\dot{x}(0) = \dot{y}(0) = -795.774715\, m/s$. For the 1st time interval, these initial conditions lead to particle standing multi-frequency wave motion, and for the considered excitation frequencies with a period of 1s.

In Figure 2, the displacements are depicted, whereas the numerically determined displacements of the two masses seem to coincide with the analytical solution obtained for the 1st time interval from equation (14) and the 2nd interval from equation (17). Therefore, the motion is a rigid body and in the augmented perpetual manifolds. The motion is entirely different in the two-time intervals.

Examining the maximum of the maxima of absolute differences between the analytical solution with all timeseries of the numerically determined displacements, for the first (second) time interval is $1.62 \cdot 10^{-9} m$ $\left(3.22 \cdot 10^{-9} m\right)$, leads to the conclusion that they are minimal.

Figure 3 depicts the velocities, whereas the velocities of the two masses seem that they coincide with the analytical solution obtained for the 1st time interval from equation (13) and for the 2nd interval from equation (16). The perfect agreement can be certified further considering the maximum of the maxima of the absolute differences between all timeseries of all the numerically determined velocities with the analytical solution for the 1st (2nd) time interval have the minimal value of $2.92 \cdot 10^{-9} m/s$ $\left(2.60 \cdot 10^{-8} m/s\right)$.

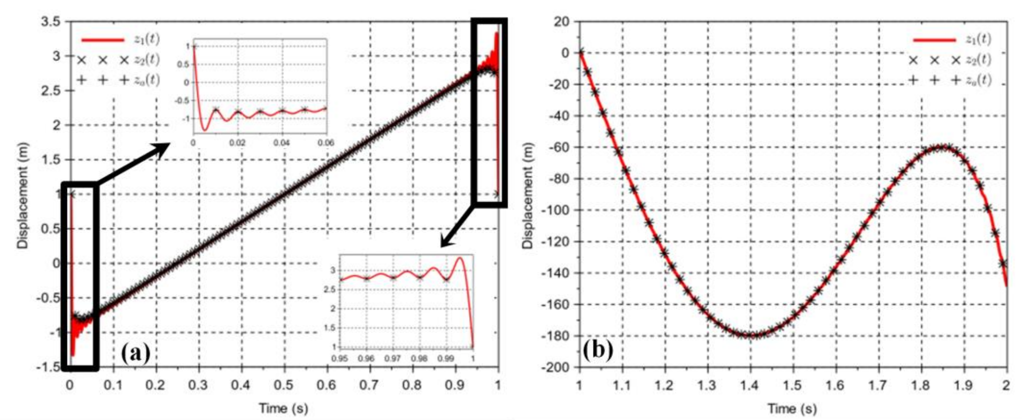

Fig. 2. Displacements of the system. (**a**) for the 1st time interval 0-1 s, and (**b**) for the 2nd time interval 1-2 s.

Therefore, the numerical results coincide with the analytical solutions derived in the previous section, which certifies the theory's validity.

Standing wave motion for the first interval can be certified by the close-ups in Figures 2a,3a, whereas the periodicity of the motion with 1s period is evident.

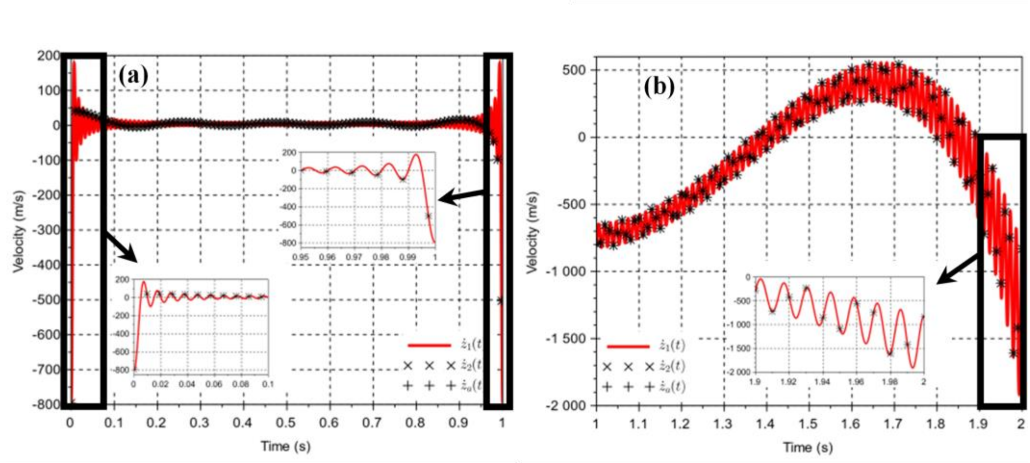

Fig. 3. Velocities of the system. (**a**) for the 1st time interval 0-1 s, and (**b**) for the 2nd time interval 1-2 s.

4. Conclusions

A corollary is written and it proved that a mechanical system's exact augmented perpetual manifolds are not unique and infinite. The theory applied in an example of a mechanical system with two different types of excitation forces and with numerical simulations is certified. The exact augmented perpetual manifold comprised of the two different ones on each time interval is a nonsmooth function with discontinuity of the vector field on the 1s that the external forces are changed. As a continuation of this work, further mathematical analysis of the mathematical space of the functions defining the exact augmented perpetual manifolds can be done, noting that they might be nonsmooth functions too, which leads to much-complicated analysis.

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