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A COMPARATIVE STUDY OF NEW HOMOTOPY PERTURBATION METHOD AND FINITE DIFFERENCE METHOD FOR SOLVING UNSTEADY HEAT CONDUCTION EQUATION

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Abstract

Calendering is a mechanical finishing technology used in the leather, textile, and paper industry, with the mechanism of reshaping and deforming fibers of the web to be calendered with the help of pressure and heat transfer. In this paper, a comparative study has been done to examine the flow of heat inside the calender nip using an unsteady heat conduction equation with specific initial and boundary conditions. The outcomes achieved by utilizing the new homotopy perturbation method and finite difference method have been compared with the exact solution. The achieved outcomes reveal the accuracy, effectiveness, and reliability of techniques applied. It is observed that the outcomes achieved by the new homotopy perturbation method are more accurate than those obtained by the finite difference method.

Keywords: Heat conduction, thermal diffusivity, homotopy, finite difference, calendering, nip mechanics

Mathematical Subject Classification: 35A20, 35A35, 35K05, 80M20

1. Introduction

In various fields of engineering, real-life problems are converted into mathematical models in the form of differential equations. In recent years, one of the methods used to solve such problems is Homotopy Perturbation Method (HPM) (Gupta and Kanth 2019a, 2019b, 2019c, 2021; Grover et al. 2012; He 2003). HPM was first introduced by Ji Huan He. It is used to solve various initial and boundary value problems (He 2000, 2003, 2006a).

Ji Huan He showed that various linear or non-linear problems that are complex in nature can be converted to a simple problem using HPM. A combination of homotopy and traditional perturbation method leads to a better technique, i.e. Homotopy perturbation method (He 2003, 2006b).

Biazar and Eslami introduced a technique, i.e. a new homotopy perturbation method (NHPM) (Biazar and Eslami 2011; Demir et al. 2013). The development of a proper homotopy condition and choices of initial approximation guess are the two significant stages of NHPM. The investigation discloses that appropriate homotopy can be developed by decomposition of source function with less computation efforts. NHPM is considered as the most influential technique to attain the solution near to the exact solution, depending upon the approximation. Nowadays, many researchers are utilizing this technique for finding an analytical approximate solution of various partial differential equations (Elbadri 2013, 2015; Mirzazadeh and Ayati 2016; Gupta and Kanth 2021).

Another method for solving PDE is the Finite Difference Method (FDM). In this method, a continuous partial differential equation is replaced with discrete approximations. The initial step in FDM is to define mesh points that are simply a uniform grid of spatial points. These grid points show the discrete positions in the space at which the solution is to be obtained. With the increase in the number of mesh points, the accuracy of these techniques increases (Biazar and Asayesh 2019). All the FDM techniques are closely related to each other, they only differ in stability, execution speed, and accuracy. Crank Nicolson (CN) method is the most commonly used method in solving parabolic partial differential equations. The Crank Nicolson method has significant advantages when time-accurate solutions are important. Also, this technique shows the accuracy of higher order and is stable unconditionally (Faduga et al. 2013).

Heat transfer plays an important role in the calendering system. Heat is transferred by conduction to fibers of a web in contact with heated rolls. The gloss and smoothness of the fabric will increase by increasing the temperature of the rolls. Hence, heat transformation has a prominent effect on the gloss and smoothness of the fabric. The above-discussed methods are useful in predicting the heat transfer due to conduction in calendering process when the web is inside the rolls of the calender (Gupta and Kanth 2018, 2019d, 2020; Kanth and Ray 2019). In this paper, a solution of unsteady heat conduction equation applicable to calendering heat transfer mechanism is achieved using NHPM and CN methods. The achieved outcomes are compared with the analytical solution which shows that NHPM is extremely prominent in comparison to the CN method.

2. Solution of unsteady heat conduction equation

In this section, two different heat conduction problems applicable to the calendering system are solved using NHPM and Crank-Nicolson method. The solution to these problems using NHPM and Crank-Nicolson method is shown graphically as three-dimensional plots. Also, a comparison between NHPM, Crank-Nicolson, and the analytical method is shown graphically.

2.1 Problem 1

Consider one-dimensional heat conduction equation

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial y^2} \tag{1}$$

with B.C. and I.C.

$$\begin{gathered} T(0,\tau) = 0,\ T(1,\tau) = 0 \\ T(y,0) = \sin\frac{\pi y}{2},\ 0 \le y \le 1 \end{gathered} \tag{2}$$

2.1.1 Solution of Problem 1 using NHPM

Using NHPM, homotopy is constructed for equation (1) given by (Gupta and Kanth 2019c),

$$(1-\mathrm{p})\left(\frac{\partial U}{\partial \tau}-T_0\right)+\mathrm{p}\left(\frac{\partial U}{\partial \tau}-\frac{\partial^2 U}{\partial y^2}\right)=0$$

or

$$\frac{\partial U(y,\tau)}{\partial \tau}=T_0(y,\tau)-\mathrm{p}\left(T_0-\frac{\partial^2 U}{\partial y^2}\right) \tag{3}$$

Taking $L^{-1}=\int_{\tau_0}^{\tau}(.)d\tau$ i.e. inverse operator on both sides of equation (3), then

$$U(y,\tau)=\int_0^{\tau}T_0(y,\tau)d\tau-\mathrm{p}\int_0^{\tau}\left(T_0-\frac{\partial^2 U}{\partial y^2}\right)d\tau+U(y,0) \tag{4}$$

Let the solution of the equation (4) is

$$U=U_0+\mathrm{pU}_1+\mathrm{p}^2U_2+\mathrm{p}^3U_3+\ldots \tag{5}$$

where $U_0,U_1,U_2,U_3,\ldots$ are to be determined.

Utilizing equation (5) in equation (4), then comparing the coefficients of p and equating to zero, (Gupta and Kanth 2019c),

$$\begin{aligned}
&\mathrm{p}^0:\ U_0(y,\tau)=\int_0^{\tau}T_0(y,\tau)d\tau+U(y,0)\\
&\mathrm{p}^1:\ U_1(y,\tau)=-\int_0^{\tau}\left(T_0(y,\tau)-\frac{\partial^2 U_0}{\partial y^2}\right)d\tau\\
&\mathrm{p}^2:\ U_2(y,\tau)=\int_0^{\tau}\left(\frac{\partial^2 U_1}{\partial y^2}\right)d\tau\\
&\mathrm{p}^3:\ U_3(y,\tau)=\int_0^{\tau}\left(\frac{\partial^2 U_2}{\partial y^2}\right)d\tau\\
&\bullet\\
&\bullet\\
&\bullet
\end{aligned} \tag{6}$$

and so on.

Consider initial approximation of equation (1) as

$$T_0(y,\tau)=\sum_{n=0}^{\infty}d_n(y)Q_n(\tau),\ U(y,0)=T(y,0),\ \ Q_k(\tau)=\tau^k \tag{7}$$

where $d_0(y),d_1(y),d_2(y),\ldots$ are unknown coefficients.

$$U_0(y,\tau)=\begin{pmatrix} d_0(y)\tau+d_1(y)\dfrac{\tau^2}{2} \\ +d_2(y)\dfrac{\tau^3}{3}+d_3(y)\dfrac{\tau^4}{4}+\ldots \end{pmatrix}+\sin\frac{\pi y}{2} \tag{8}$$

$$U_1(y,\tau)=\left(-d_0(y)-\frac{\pi^2}{4}\sin\frac{\pi y}{2}\right)\tau+\left(-\frac{1}{2}d_1(y)+\frac{1}{2}d_0''(y)\right)\tau^2$$
$$+\left(-\frac{1}{3}d_2(y)+\frac{1}{3}d_1''(y)\right)\tau^3+\ldots$$

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and so on.

Solving above equations satisfying $U_1(y,\tau)=0$, then from equation (8)

$$d_1(y)=-\frac{\pi^2}{4}\sin\frac{\pi y}{2},\ d_1(y)=\frac{\pi^4}{16}\sin\frac{\pi y}{2},\ d_2(y)=-\frac{\pi^6}{64}\sin\frac{\pi y}{2}$$

Therefore

$$\begin{aligned}T(y,\tau)=U_0(y,\tau)&=\sin\frac{\pi y}{2}+d_0(y)\tau+d_1(y)\frac{\tau^2}{2}+d_2(y)\frac{\tau^3}{3}+d_3(y)\frac{\tau^4}{4}+\ldots\\ &=\sin\frac{\pi y}{2}\left[1-\frac{\pi^2}{2^2}\tau+\frac{\pi^4}{2^4}\frac{\tau^2}{2}-\frac{\pi^6}{2^6}\frac{\tau^3}{3}+\ldots\right]\\ &=\sin\frac{\pi y}{2}e^{-\frac{\pi^2\tau}{4}}\end{aligned} \tag{9}$$

thus

$$T(y,\tau)=\sum_{n=1}^{\infty}\sin\frac{\pi y}{2}e^{-\frac{n^2\pi^2\tau}{4}} \tag{10}$$

Equation (10) can be used to find the temperature distribution for a web as shown in Figure 1.

2.1.2 Solution of Problem 1 using CN Method

CN method is 2nd order in time and it is unconditionally stable for transient heat conduction equation. The accuracy of this method is the same in both space and time. This method has significant advantages when time-accurate solutions are important. Therefore, the CN method is considered as an efficient technique for solving heat conduction equation under different initial and boundary conditions.

In this method, the difference formula for time derivative is

$$\frac{\partial T}{\partial \tau} = \frac{T_{i,j+1} - T_{i,j}}{\Delta \tau} + O(\Delta \tau) \tag{11}$$

And the central difference formula at time $\tau_{j+1/2}$ for the spatial derivative is

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{2}\left(\frac{T_{i-1,j+1} - 2T_{i,j+1} + T_{i+1,j+1}}{\Delta y^2} + \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta y^2} \right) + O(y) \tag{12}$$

Using equations (11) and (12), equation (2) can be rewritten as

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta \tau} = \frac{1}{2}\alpha \left(\frac{T_{i-1,j+1} - 2T_{i,j+1} + T_{i+1,j+1}}{\Delta y^2} + \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta y^2} \right) \tag{13}$$

On rearranging,

$$T_{i,j+1} - T_{i,j} = \frac{1}{2}\alpha \frac{\Delta \tau}{(\Delta y)^2}\left(T_{i-1,j+1} - 2T_{i,j+1} + T_{i+1,j+1} + T_{i-1,j} - 2T_{i,j} + T_{i+1,j} \right) \tag{14}$$

Here r is the dimensionless diffusion number given by

$$r = \alpha \frac{\Delta \tau}{(\Delta y)^2} \tag{15}$$

Using equation (15),

$$-rT_{i-1,j+1} + (2+2r)T_{i,j+1} - rT_{i+1,j+1} = rT_{i-1,j} + (2-2r)T_{i,j} + rT_{i+1,j} \tag{16}$$

The above equation is solved using MATLAB and outcomes are shown graphically in Figure 2.

2.2 Problem 2

Consider one-dimensional heat conduction equation

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial y^2} \tag{17}$$

with B.C. and I.C.

$$\begin{gathered} T(0,\tau) = 0,\ T(1,\tau) = 0 \\ T(y,0) = \sin \pi y,\ 0 \le y \le 1 \end{gathered} \tag{18}$$

2.2.1 Solution of Problem 2 using NHPM

Using NHPM, homotopy is constructed for equation (1) given by (Gupta and Kanth 2019c),

$$\left(1-\mathrm{p}\right)\left(\frac{\partial U}{\partial \tau}-T_0\right)+\mathrm{p}\left(\frac{\partial U}{\partial \tau}-\frac{\partial^2 U}{\partial y^2}\right)=0$$

or

$$\frac{\partial U(y,\tau)}{\partial \tau}=T_0(y,\tau)-\mathrm{p}\left(T_0-\frac{\partial^2 U}{\partial y^2}\right) \tag{19}$$

Taking $L^{-1}=\int_{\tau_0}^{\tau}(.)d\tau$ i.e. inverse operator on both sides of equation (14), then

$$U(y,\tau)=\int_0^{\tau} T_0(y,\tau)d\tau-\mathrm{p}\int_0^{\tau}\left(T_0-\frac{\partial^2 U}{\partial y^2}\right)d\tau+U(y,0) \tag{20}$$

Let the solution of the equation (20) is

$$U=U_0+\mathrm{pU}_1+\mathrm{p}^2U_2+\mathrm{p}^3U_3+\ldots \tag{21}$$

where $U_0,U_1,U_2,U_3,\ldots$ are to be determined.

Utilizing equation (21) in equation (20), then comparing the coefficients of p and equating to zero,

$$\begin{aligned}
&\mathrm{p}^0:\ U_0(y,\tau)=\int_0^{\tau} T_0(y,\tau)d\tau+U(y,0)\\
&\mathrm{p}^1:\ U_1(y,\tau)=-\int_0^{\tau}\left(T_0(y,\tau)-\frac{\partial^2 U_0}{\partial y^2}\right)d\tau\\
&\mathrm{p}^2:\ U_2(y,\tau)=\int_0^{\tau}\left(\frac{\partial^2 U_1}{\partial y^2}\right)d\tau\\
&\mathrm{p}^3:\ U_3(y,\tau)=\int_0^{\tau}\left(\frac{\partial^2 U_2}{\partial y^2}\right)d\tau\\
&\quad\bullet\\
&\quad\bullet\\
&\quad\bullet
\end{aligned} \tag{22}$$

and so on.

Consider initial approximation of equation (18) as

$$T_0(y,\tau)=\sum_{n=0}^{\infty} d_n(y)Q_n(\tau),\ U(y,0)=T(y,0),\ \ Q_k(\tau)=\tau^k \tag{23}$$

where $d_0(y), d_1(y), d_2(y),\ldots$ are unknown coefficients.

Utilizing equation (23) in equation (22),

$$U_0(y,\tau)=\left(d_0(y)\tau+d_1(y)\frac{\tau^2}{2}+d_2(y)\frac{\tau^3}{3}+d_3(y)\frac{\tau^4}{4}+\ldots\right)+\sin\pi y \tag{24}$$

$$U_1(y,\tau)=\left(-d_0(y)-\pi^2\sin\pi y\right)\tau+\left(-\frac{1}{2}d_1(y)+\frac{1}{2}d_0''(y)\right)\tau^2$$
$$+\left(-\frac{1}{3}d_2(y)+\frac{1}{3}d_1''(y)\right)\tau^3+\ldots$$

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and so on.

On solving the above equations satisfying, then from equation (24)

$$d_0(y)=-\pi^2\sin\pi y,\ d_1(y)=\pi^4\sin\pi y,\ d_2(y)=-\pi^6\sin\pi y$$

Therefore

$$\begin{aligned}T(y,\tau)=U_0(y,\tau)&=\sin\pi y+d_0(y)\tau+d_1(y)\frac{\tau^2}{2}+d_2(y)\frac{\tau^3}{3}+d_3(y)\frac{\tau^4}{4}+\ldots\\&=\sin\pi y\left[1-\pi^2\tau+\pi^4\frac{\tau^2}{2}-\pi^6\frac{\tau^3}{3}+\ldots\right]\\&=\sin\pi y e^{-\pi^2\tau}\end{aligned} \tag{25}$$

thus

$$T(y,\tau)=\sum_{n=1}^{\infty}\sin\pi y e^{-n^2\pi^2\tau} \tag{26}$$

Equation (10) can be used to find the temperature distribution for a web as shown in Figure 4.

2.1.2 Solution of Problem 2 using CN Method

Equation (17) under B.C. and I.C. given by equation (18) is solved with the help of the CN Method using equation (16) as discussed in problem 1. The outcomes obtained using MATLAB are shown graphically in Figure 5.

3. Results and Discussion

The solution of problem 1 by utilizing the NHPM and CN method is shown graphically as three-dimensional plots in Figure 1 and Figure 2, respectively. Also, the comparison of NHPM, CN, and the analytical methods has been done and shown graphically in Figure 3 which clearly indicates that the solution achieved utilizing NHPM is the same as the solution achieved using the analytical method, while outcomes obtained using the CN method are not exactly the same. Therefore, NHPM shows better outcomes as compared to the CN method.

Similarly, the solution of problem 2 by utilizing NHPM and CN method is shown graphically as three-dimensional plots in Figure 4 and Figure 5, respectively. Also, the comparison of NHPM and CN method has been done and shown graphically in Figure 6, which clearly indicates that the solution achieved utilizing NHPM is the same as the solution achieved using the analytical method, while outcomes obtained using the CN method are not exactly the same. Therefore, NHPM shows better outcomes as compared to the CN method.

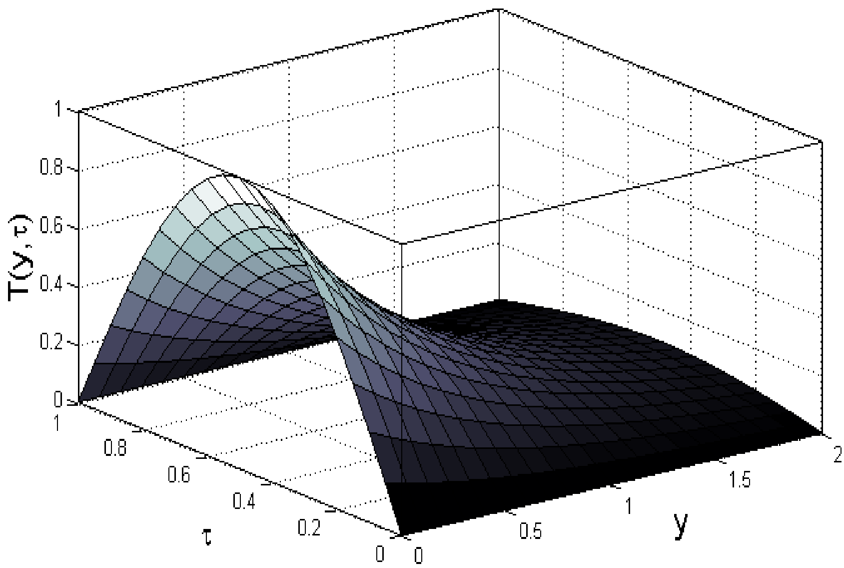

Fig. 1. Solution of Problem 1 utilizing NHPM.

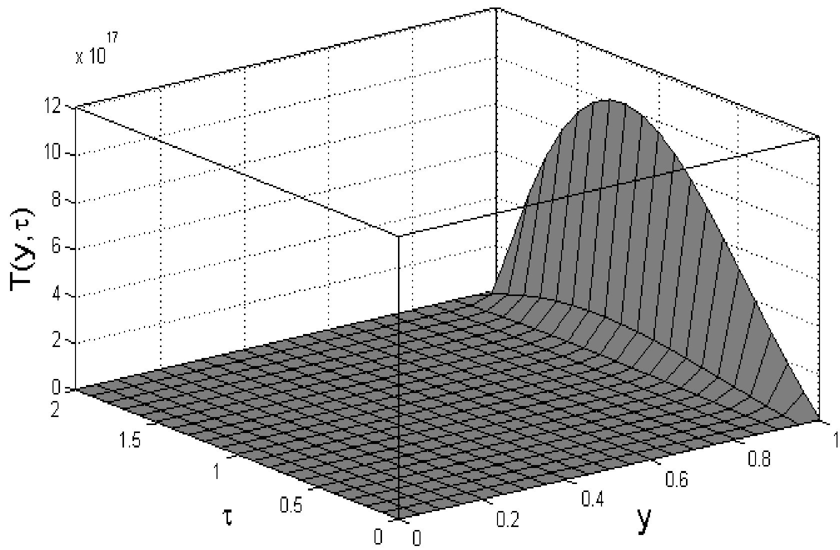

Fig. 2. Solution of Problem 1 utilizing the CN method.

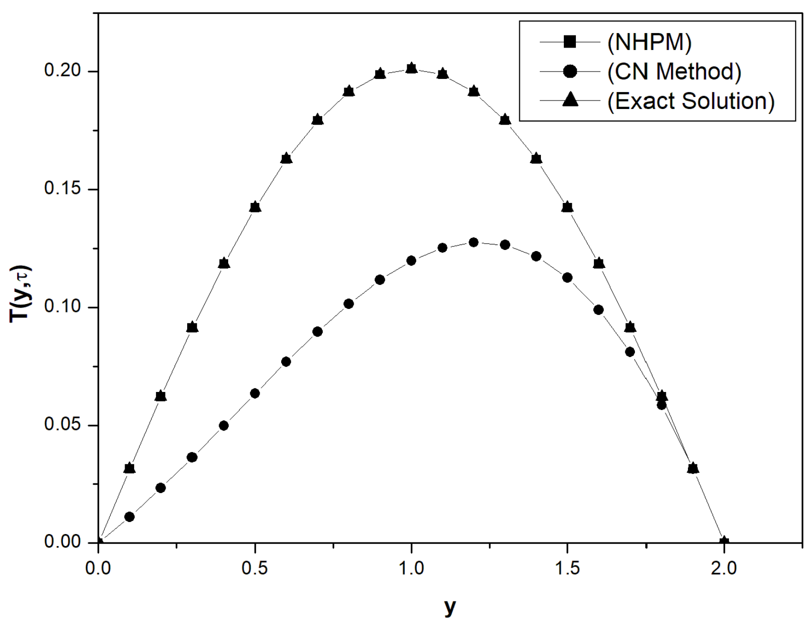

Fig. 3. Comparison of NHPM, CN method, and separation of variable method for problem 1.

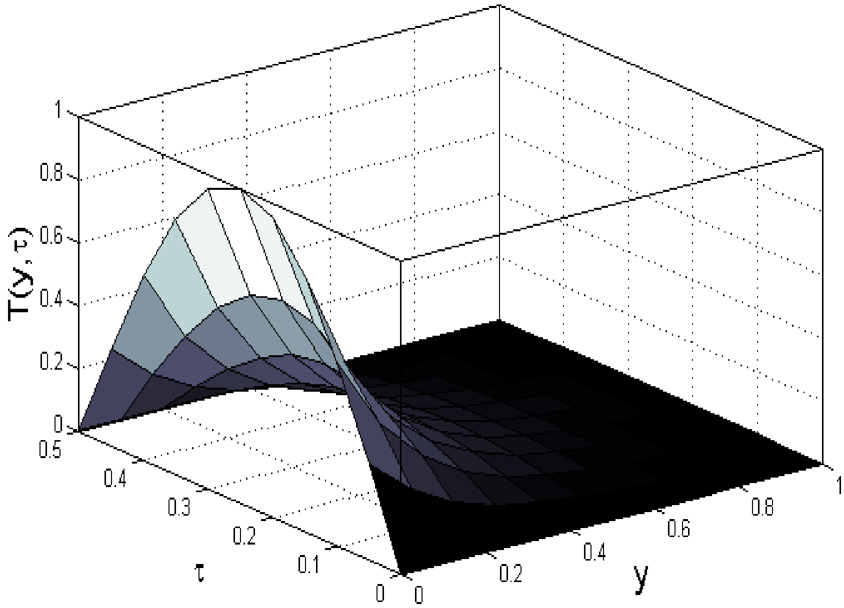

Fig. 4. Solution of Problem 2 utilizing NHPM.

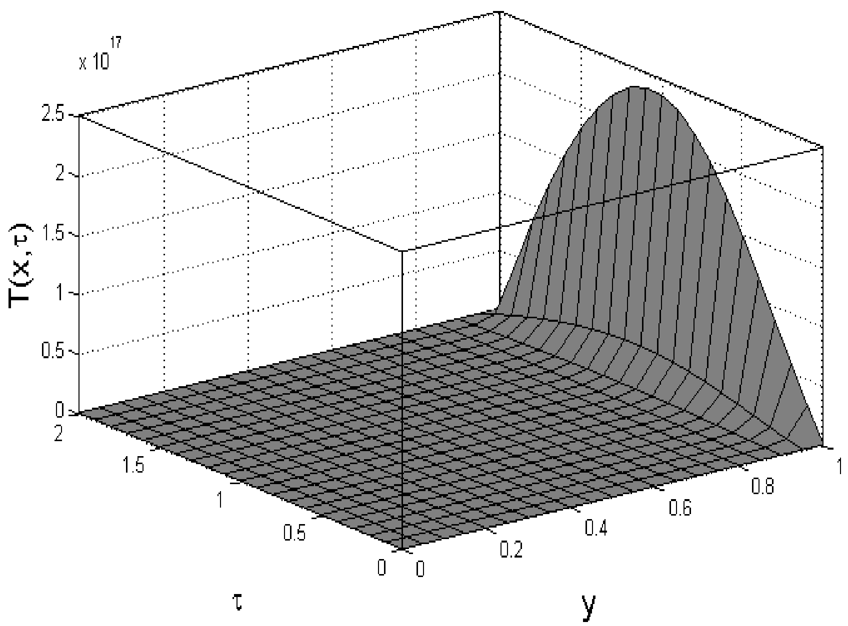

Fig. 5. Solution of Problem 2 utilizing CN method.

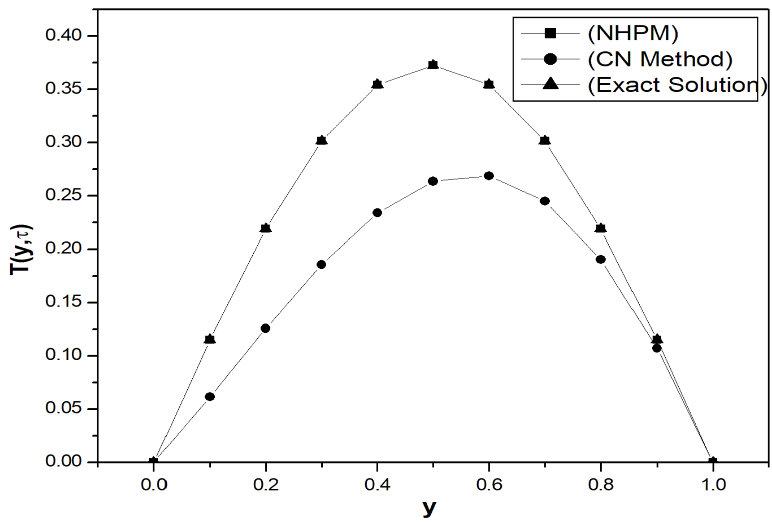

Fig. 6. Comparison of NHPM, CN method, and separation of variable method for problem 2.

4. Conclusion

The approximate analytical and numerical solutions of 1D heat equation under different initial and boundary conditions suitable for calendering process are achieved by utilizing NHPM and CN techniques. The outcomes of both techniques are compared, which shows that NHPM gives the outcomes that match completely with the exact results while the results obtained using the CN method show errors. Therefore, NHPM has remarkable applicability for finding temperature profiles during the roll-to-roll contact problem of the calendering system used in various industries.

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