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A NUMERICAL ALGORITHM BASED ON THE RCW METHOD TO SOLVE A SET OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

This research paper deals with a numerical algorithm based on the RCW method to solve a set of first-order ordinary differential equations. In this algorithm, the answer of each equation is considered as a polynomial of degree 2. The coefficients used in this polynomial are categorized into groups of free and fixed coefficients. The free coefficients are obtained by optimizing the error function, while the fixed coefficients are computed from the magnitudes of derivative of this polynomial at initial value. To check the correctness of the current algorithm, its results for several case problems are compared with the findings of the Runge-Kutta (RK) method. This comparison shows that the algorithm presented in this paper has a higher accuracy and stability than the RK method even for the nonlinear problems.

Keywords: RCW method, ordinary differential equations, initial value problems, first-order equations

1. Introduction

Many physical phenomena, chemical reactions, mechanical systems, thermal systems and engineering applications can be simulated using the ordinary and partial differential equations (Zeeshan and Majeed 2016, Jaskulski et al. 2017, Atashafrooz et al. 2018, Reis et al. 2018, Sajjadi et al. 2018, Shah et al. 2018, Mahmoodabadi et al. 2018, Mehralian and Beni 2018, Kumar et al. 2018, Atashafrooz et al. 2019, Kang et al. 2019, Sajjadi et al. 2019, Rashid et al. 2019, Sheikholeslami et al. 2019, Atashafrooz and Asadi 2019).

In many cases, partial differential equations (PDEs) can be converted into a set of ordinary differential equations (ODEs) using spetial discretization (Esmaeilpour and Ganji 2007, Blinder 2013). Also, there are approaches to reduce the order of ordinary differential equations and to their change to a set of first-order ODEs (Ha 2001, Charroyer et al. 2018, Filipov et al. 2019). Therefore, finding an approach to solve a set of first-order ODEs is very important and practical.

So far, many scholars have focused on this topic and several different algorithms were presented to solve these equations, such as Euler's method, Taylor series algorithms, Runge-Kutta methods, improved Euler's method, multi-step approaches and the extrapolation method

(Franco 2007, Brugnano and Magherini 2009, Cutolo et al. 2011, Tang and Sun 2012, Li and Wu 2016, Wang et al. 2017, Rahmanzadeh and Barfeie 2018, Korkmaz 2019, Amodio et al. 2019).

RCW method is an effective approach to solve an ordinary differential equation. This algorithm was firstly presented by Rahmanzadeh et al. (2013). Recently, RCW method was applied to simulate the Blasius problem (Rahmanzadeh et al. 2020). This problem simulates the viscous fluid flow over a semi-infinite flat plate and includes a nonlinear ordinary differential equation.

However, previous research has clearly shown that the RCW method has a high ability to solve the ODE problems. Therefore, the authors of the current research have decided to extend this method to solve a system of first-order initial-value problems. In other words, the main goal of this paper is to present a numerical algorithm based on the modified RCW method to solve a set of first-order ordinary differential equations. In the next sections, in addition to presenting the "Theory" and "Results and Discussions" parts, several case problems are considered and solved to ensure the correctness of the presented algorithm.

2. Theory

Consider a set of first-order differential equations as follows:

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(t, y_1, y_2, \cdots, y_m) \qquad , y_1(t_0) = y_{01} \\ \frac{dy_2}{dt} &= f_2(t, y_1, y_2, \cdots, y_m) \qquad , y_2(t_0) = y_{02} \\ &\vdots \\ \frac{dy_m}{dt} &= f1(t, y_1, y_2, \cdots, y_m) \qquad , y_m(t_0) = y_{0m} \end{aligned} \tag{1}$$

To solve this system using RCW method, it is assumed that the solution of an equation in the above set is as follows:

$$y_i = \frac{a_i}{2}(t-t_0)^2 + a_{0i}(t-t_0) + y_{01} \quad , a_{0i} = f_i(t_0, y_{01}, \cdots, y_{0m}) \tag{2}$$

Or

$$y_i' = a_i(t-t_0) + a_{0i} \quad , i = 1, 2, \cdots, m \tag{3}$$

Here, the a_i variables $(i = 1, 2, \cdots, m)$ are named as the free coefficients. To calculate these coefficients, it is first necessary to define the residual functions (R_i) as follows:

$$\begin{aligned} R_1 &= y_1' - f_1(t, y_1, y_2, \cdots, y_m) \\ R_2 &= y_2' - f_2(t, y_1, y_2, \cdots, y_m) \\ &\vdots \\ R_m &= y_m' - f_m(t, y_1, y_2, \cdots, y_m) \end{aligned} \tag{4}$$

Then, an error function (e_i) is defined based on these residual functions using the following equation:

$$e_i = \int_{t_0}^{t_0+h} R_i^2 dt \tag{5}$$

The next step to calculate the free coefficients is the optimization of this error function. This optimization is performed as follows:

$$\begin{aligned}
de_1(a_1, a_2, \cdots, a_m) &= \int_{t_0}^{t_0+h} R_1 \cdot \frac{\partial R_1}{\partial a_1} dt = 0 \\
de_2(a_1, a_2, \cdots, a_m) &= \int_{t_0}^{t_0+h} R_2 \cdot \frac{\partial R_2}{\partial a_2} dt = 0 \\
&\vdots \\
de_m(a_1, a_2, \cdots, a_m) &= \int_{t_0}^{t_0+h} R_m \cdot \frac{\partial R_m}{\partial a_m} dt = 0
\end{aligned} \tag{6}$$

In fact, the values of $a_1, a_2, \cdots$ and a_m are found by solving the above equations.

Besides, a_{0i} variables in Equations (2) and (3) are called fixed coefficients and based on the Equation set (1) their magnitudes are obtained from the values of y_i' at the t_0 point.

However, by replacing the fixed and free coefficients in Equation (2), the final answer of Equation system (1) can be computed as follows:

$$y_i(t_0 + h) = \int_{t_0}^{t_0+h} f_i(t, y_1, y_2, \cdots, y_m) \cdot dt \qquad \text{i=1,2,}\cdots\text{,m} \tag{7}$$

Accordingly, more details on the current algorithm needed to solve a set of first-order ordinary differential equations are presented as follows:

1- The matrix F is defined and calculated as follows:

$$F = \begin{bmatrix} F_1 & F_2 & \cdots & F_m \end{bmatrix} \tag{8}$$

Where

$$F_i = (t - t_0) \times \left(\frac{(t - t_0)}{2} \times \frac{\partial f_i}{\partial y_i} - 1 \right) \tag{9}$$

2- The matrix E is formed in the following manner:

$$E = \begin{bmatrix} E_1 & E_2 & \cdots & E_m \end{bmatrix} = 0 \tag{10}$$

Here

$$E_i = R_i \cdot F_i = 0 \tag{11}$$

In fact, the above equation leads to the formation of m algebraic equations. Such that the free coefficients (a_i) are obtained by solving these algebraic equations using the Newton method.

3- To calculate the values of $y_i(t_0 + h)$, the following steps are carried out:

$$t_{0k} = t_0 + (k - 1)\frac{h}{4} \quad , \quad k = 1, \cdots, 4 \tag{12}$$

$$y_{0i1} = y_{0i} \tag{13}$$

$$a_{0ik} = f_i(t_{0k}, y_{01k}, y_{02k}, \cdots, y_{0mk}) \tag{14}$$

$$y_i' = a_i(t - t_{0k}) + a_{0ik} \tag{15}$$

$$y_i = \frac{a_i}{2}\left(t - t_{0k}\right)^2 + a_{0ik}\left(t - t_{0k}\right) + y_{0ik} \tag{16}$$

$$J = Jacob\left(\left[E_1, E_2, \cdots, E_m\right], \left[a_1, a_2, \cdots, a_m\right]\right) \tag{17}$$

Here, $J = Jacob\left(\left[E_1, E_2, \cdots, E_m\right], \left[a_1, a_2, \cdots, a_m\right]\right)$ is the Jacobin of E_i against a_i variables $\left(i = 1, 2, \cdots, m\right)$, which can be calculated as:

$$J = Jacob\left(\left[E_1, E_2, \cdots, E_m\right], \left[a_1, a_2, \cdots, a_m\right]\right) = \begin{bmatrix} \frac{\partial E_1}{\partial a_1} & \cdots & \frac{\partial E_1}{\partial a_m} \\ \cdots & \ddots & \\ \frac{\partial E_m}{\partial a_1} & \cdots & \frac{\partial E_m}{\partial a_m} \end{bmatrix} \tag{18}$$

By guessing the values of $a_1, a_2, \cdots, a_m$ and integrating of all arrays of matrices E and J relative to t variable in the interval of $\left[t_0 \;\; t_0 + h\right]$, the new magnitudes of $a_i \left(i = 1, 2, \cdots, m\right)$ will be calculated as follows:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} - \left[\int_{t_0}^{t_0+h} J dt\right]^{-1} \cdot \int_{t_0}^{t_0+h} E \; dt \tag{19}$$

These calculations continue until the a_i variables are converged. With this strategy, the values of y_{0ik+1} are obtained using the following equation:

$$y_{0ik+1} = \int_{t_{0k}}^{t_{0k}+h/4} f_i(t, y_1, y_2, \cdots, y_m) \cdot dt + y_{0ik} \tag{20}$$

Finally, the value of y_{0i5} is the values of $y_i(t_0 + h)$.

3. Results and Discussions

In this section, the presented steps for solving a set of first-order ordinary differential equations using the RCW method are discussed in the form of a simple example.

Consider the following first-order ordinary differential equation:

$$y_1' = -20 y_1 \;,\; y_1(0) = 1 \tag{21}$$

First, function F and matrix E are calculated using the below equations:

$$F = -10\left(t - t_0\right)^2 - \left(t - t_0\right) \tag{22}$$

$$E = R_1 \cdot \left[\left(t - t_0 \right) + 10 \left(t - t_0 \right)^2 \right] \tag{23}$$

Where

$$R_1 = 20 y_1 + y_1' \tag{24}$$

Then, value of k set to one, such that the fixed coefficients are computed using the following equations:

$$t_{0k} = t_{01} = t_0 = 0 \tag{25}$$

$$y_{01k} = y_{011} = y_{01} = 1 \tag{26}$$

$$a_{011} = f(t_{01}, y_{011}) = -20 \tag{27}$$

Therefore,

$$y_1' = a_1 \left(t - t_{01} \right) - a_{011} = a_1 \cdot t - 20 \tag{28}$$

$$y_1 = \frac{a_1}{2} t^2 - 20 \cdot t + 1 \tag{29}$$

In the next step, the magnitudes of y and y' are replaced in matrix E, such that the values of this matrix are dependent on the a_1 and t variables.

Then, matrix J is formed in the following manner:

$$J = jacob\left(E, [a_1] \right) = \left(10t^2 + t \right)^2 \tag{30}$$

Then, an arbitrary value is guessed for a_1 variable. With this strategy, matrixes E and J are dependent on the t variable. By Integrating over t variable in interval $[t_0 \;\; \mathrm{t}_0 + h]$, the new value of a_1 is numerically calculated as:

$$a_1 = a_1 - \left[\int_0^{0.05} J \cdot dt \right]^{-1} \cdot \int_0^{0.05} E \cdot dt \tag{31}$$

After the convergence of the numerical solution, the final value of a_1 is equal to. 289.4737. By substituting the a_1 in function y_1, the magnitude of y_{012} is obtained as:

$$y_{01k+1} = y_{012} = \int_0^{0.05/4} -20 y_1 \cdot dt + y_{011} = 0.7794 \tag{32}$$

In next step, the value of k set to 2 and the new coefficients are calculated as follows:

$$t_{02} = t_0 + \left(k - 1 \right) \frac{h}{4} = \frac{0.05}{4} \tag{33}$$

$$y_{01k} = y_{012} = 0.7794 \tag{34}$$

$$a_{012} = f_i \left(t_{02}, y_{012} \right) = -15.5873 \tag{35}$$

$$y_1' = a_1 \left(t - t_{02} \right) - 15.5873 \tag{36}$$

$$y_1 = a_1\left(t - t_{02}\right) - 15.5873\left(t - t_{02}\right) + 0.7794 \quad (37)$$

Then, y_{013} can be calculated like the $k = 1$ step:

$$y_{013} = y_{012} + \int_{0.05/4}^{0.05/2} f_1\left(t, y_1\right) dt = 0.6073 \quad (38)$$

Then, the calculations continue for $k = 3$ and $k = 4$.

According to what is done in $k = 1$ to $k = 4$ steps, the value of y_{015} is obtained. In fact, the final value of y_1 is equal to y_{015}.

In the following, an attempt is made to show the influences of k on the correctness of numerical solution.

The analytic solution of Equation (21) is:

$$y_{exact}(t) = \exp(-20t) \quad (39)$$

Such that

$$y'_{exact}(t) = -20\exp(-20t) \quad (40)$$

The values of y'_{exac} and y'_{1k} (k=1,2,3,4) are shown in Figure 1. It should be mentioned that y'_{1k} are estimated by a linear equation.

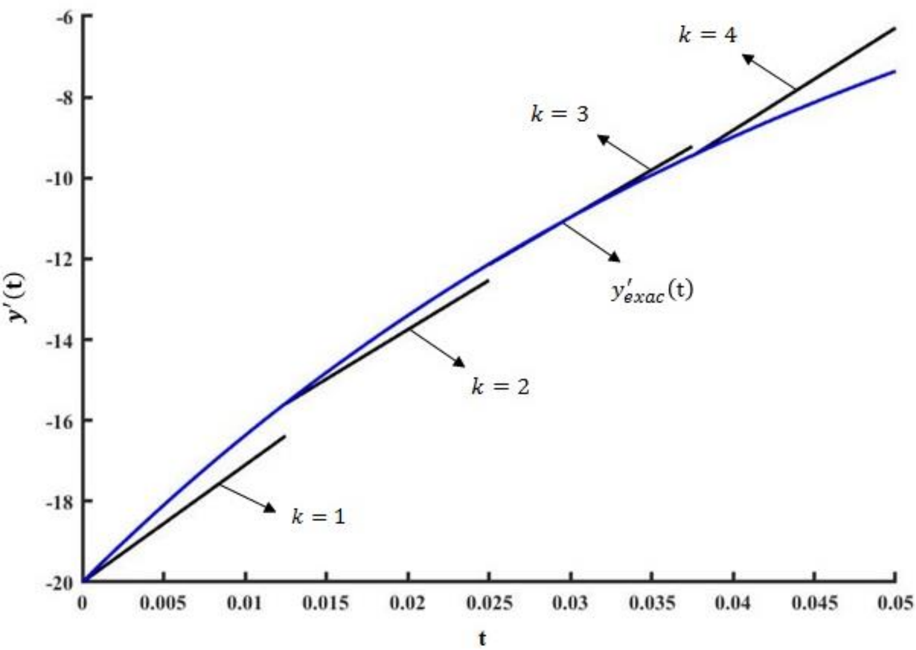

Fig. 1. The values of y'_{exac} and y'_{1k} (k=1,2,3,4)

As it is shown in Figure 1, the values of y'_{11} and y'_{12} are lower than $y'_{exac}(t)$, whilst y'_{13} and y'_{14} have higher values than $y'_{exac}(t)$.

Given that error values between the exact solution (y'_{exac}) and the estimated solutions (y'_{1k}) can be positive and negative, it can be concluded that the values of final error at the desired point $(t_0 + h)$ is decreased by summing the positive and negative errors.

However, it can be seen from Figure 2 that the absolute magnitude of error is very low for $k = 4$.

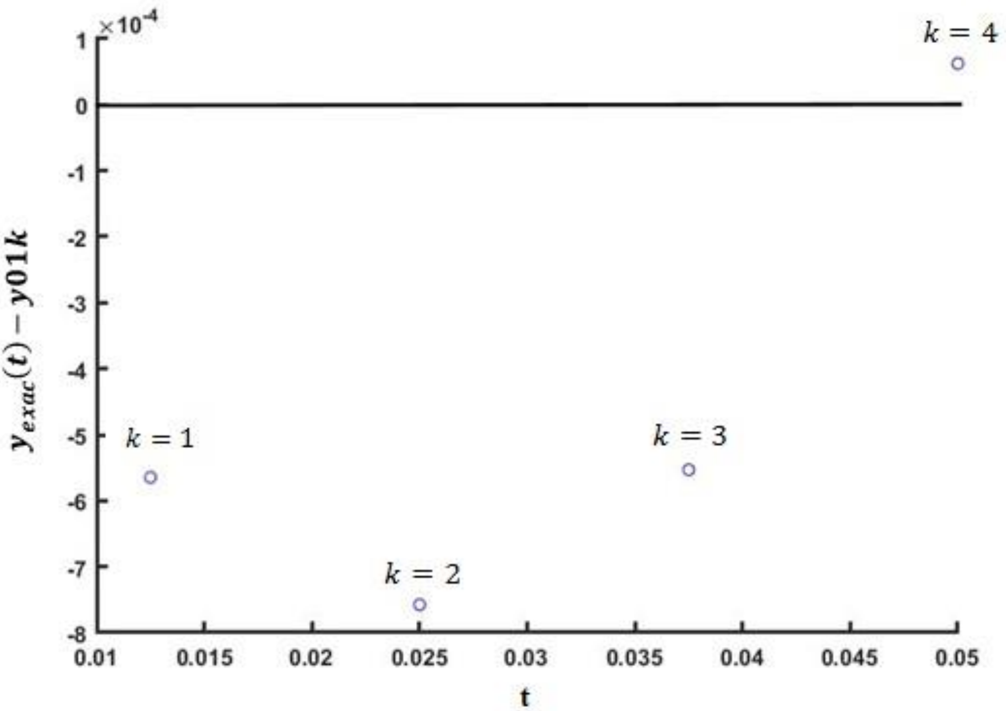

Fig. 2. The error magnitudes for different values of k

4. Examples

In this section, three different problems are selected to ensure the accuracy of the present approach for solving a set of first-order ordinary differential equations. In all the problems, the equations system is solved using the RCW and Runge-Kutta (RK) methods. In fact, these solutions are performed to compare the error values of these two methods.

4.1. Problem I

Consider the following equations system:

$$y_1' = 9y_1 + 24y_2 + 5\cos(t) - \frac{1}{3}\sin(t), \qquad y_1(0) = \frac{4}{3} \tag{41}$$

$$y_2' = -24y_1 - 51y_2 - 9\cos(t) + \frac{1}{3}\sin(t), \qquad y_2(0) = \frac{2}{3} \tag{42}$$

The errors magnitudes of RCW and RK methods in computing y_1 and y_2 variables are presented in Table 1. These results are obtained for the case of $h = 0.02$. As it can be seen from this table, the RCW method is more accurate than the RK method.

t	RCW method		RK method	
	Error of y_1	Error of y_2	Error of y_1	Error of y_2
0.02	2.54396×10^{-5}	-6.61495×10^{-5}	0.002125136	-0.00425029
0.04	1.83852×10^{-5}	-5.81536×10^{-5}	0.001952972	-0.003905981
0.06	6.85835×10^{-6}	-3.70662×10^{-5}	0.001346125	-0.002692301
0.08	-1.95706×10^{-6}	-1.95492×10^{-5}	0.000824814	-0.001649691
0.1	-7.36233×10^{-6}	-8.04911×10^{-6}	0.000473872	-0.000947818

Table 1. The errors magnitudes of RCW and RK methods in computing y_1 and y_2 variables for Problem I

4.2. Problem II

Consider a set of nonlinear ordinary differential equations as below:

$$\frac{dy_1}{dt} = \frac{3y_1y_2}{t^3} + y_1^2 - e^{6t}, \qquad y_1(1)=e^3 \tag{43}$$

$$\frac{dy_2}{dt} = 3y_2^{\frac{2}{3}} + \log(y_1) - 3t, \qquad y_2(1)=1 \tag{44}$$

Where the exact solutions are $y_{1,exact}(t) = e^{3t}$ and $y_{2,exact}(t) = t^3$.

The amounts of y_1 and y_2 obtained from the RCW and RK methods are shown in Figure 3 for the case of $h = 0.02$. These findings are compared with the results of exact solutions.

The detailed analysis of this figure clearly shows that the accuracy and stability of the RCW method is higher than the RK method. In fact, the RK method loses its stability at the point of $t = 1.16$.

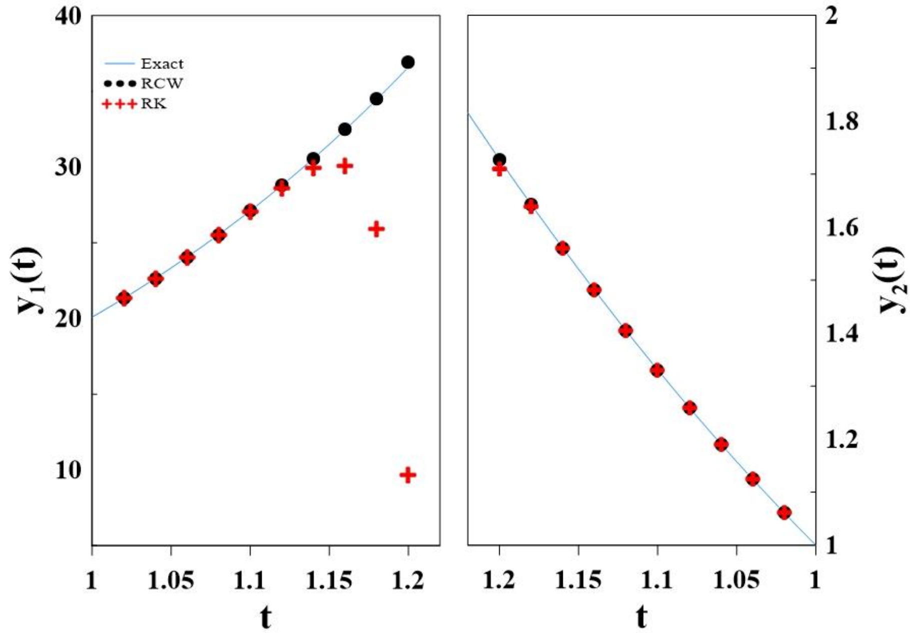

Fig. 3. Comparison of y_1 and y_2 amounts obtained from the RCW and RK methods with the results of the exact solutions for Problem II

4.3. Problem III

Consider a set of nonlinear, time dependent ordinary differential equations as follows:

$$\frac{dy_A}{dt} = -k_1 y_A y_B, \qquad y_{A0} = 55 \tag{45}$$

$$. \quad . \tag{46}$$

$$\frac{\mathrm{dy_R}}{dt} = k_1 y_A y_B - k_2 y_B y_R, \qquad y_{R0} = 0 \tag{47}$$

$$\frac{\mathrm{dy_s}}{dt} = k_2 y_B y_R, \qquad y_{s0} = 0 \tag{48}$$

Here, y_A, y_B, y_R and y_s variables point to the concentration of various materials, while y_{A0}, y_{B0}, y_{R0} and y_{s0} parameters are initial concentration of these materials. In fact, this set of equations is related to chemical reactions in a batch reactor.

The values of relative error for y_A variable in both RCW and RK methods are presented in Table 2. These values are tabulated for different values of h. As it is seen from this table, the accuracy of the RK method decreases considerably by increasing the amounts of h. In fact, Table 2 clearly shows that the stability of the RK method is not suitable at high values of h. However, the accuracy and stability of the RCW method for all presented step lengths is very good.

h	Relative error (%) for RCW	Relative error (%) for RK

0.006	0.191091	1.848575
0.007	0.33683	6.235279
0.008	0.555213	17.00306
0.009	0.876214	40.20436
0.01	1.346192	85.41022

Table 2. Values of relative error for y_A variable in both RCW and RK methods for Problem III

5. Conclusion

In this research, an attempt is made to present a numerical approach based on the RCW method to solve a set of first-order ordinary differential equations. To reach this goal, a polynomial of degree 2 is considered as the answer of each equation. This polynomial includes two groups of free and fixed coefficients. The fixed coefficients are obtained from the derivative magnitudes of this polynomial at initial values, while the free coefficients are computed by optimizing the error function. To provide more details about the mentioned algorithm, all the steps of the numerical solution are described in the form of a sample example. Besides, the findings of the current algorithm for three case problems are compared with the results of the Runge-Kutta method. This comparison clearly shows that the approach used in this paper has a higher stability and accuracy than the Runge-Kutta method even for the nonlinear problems. In addition, the results of the RCW method are in an excellent agreement with the findings of the exact solution.

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